



DEPARTMENT OF EDUCATION

GRADE 12 MATHEMATICS A

12.1: PATTERNS AND ALGEBRA



FODE DISTANCE LEARNING



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PAPUA NEW GUINEA



GRADE 12

MATHEMATICS A

UNIT MODULE 3

CALCULUS

TOPIC 1: LIMITS AND CONTINUITY

TOPIC 2: THE DERIVATIVES

TOPIC 3: APPLICATION OF DERIVATIVES

TOPIC 4: INTEGRAL CALCULUS



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MR. DEMAS TONGOGO
Principal-FODE



Flexible Open and Distance Education
Papua New Guinea

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SECRETARY'S MESSAGE

Achieving a better future by individuals students, their families, communities or the nation as a whole, depends on the curriculum and the way it is delivered.

This course is part and parcel of the new reformed curriculum. Its learning outcomes are student centred and written in terms that allow them to be demonstrated, assessed and measured.

It maintains the rationale, goals, aims and principles of the National OBE Curriculum and identifies the knowledge, skills, attitudes and values that students should achieve.

This is a provision of Flexible, Open and Distance Education as an alternative pathway of formal education.

The Course promotes Papua New Guinea values and beliefs which are found in our constitution, Government policies and reports. It is developed in line with the National Education Plan (2005 – 2014) and addresses an increase in the number of school leavers which has been coupled with a limited access to secondary and higher educational institutions.

Flexible, Open and Distance Education is guided by the Department of Education's Mission which is fivefold:

- to facilitate and promote integral development of every individual
- to develop and encourage an education system which satisfies the requirements of Papua New Guinea and its people
- to establish, preserve, and improve standards of education throughout Papua New Guinea
- to make the benefits of such education available as widely as possible to all of the people
- to make education accessible to the physically, mentally and socially handicapped as well as to those who are educationally disadvantaged

The College is enhanced to provide alternative and comparable pathways for students and adults to complete their education, through one system, many pathways and same learning outcomes.

It is our vision that Papua New Guineans harness all appropriate and affordable technologies to pursue this program.

I commend all those teachers, curriculum writers and instructional designers, who have contributed so much in developing this course.



.....
DR. ULE KOMBRA, PhD
Acting Secretary for Education



UNIT 3: CALCULUS

Introduction

Calculus is built upon two great ideas, the **derivative**, which examines slopes of a curve and rates of change and the **integral**, which deals with sums, areas, and volumes. Both of these ideas are the basis for much of mathematics and for many accomplishments towards the new millennium.

Calculus is the branch of mathematics that deals with the analysis of movement and change.

Calculus was first developed in the seventeenth century by two mathematicians working independently of each other.

Gottfried Leibnitz, a German published his work on calculus in 1684 while Isaac Newton of England published his own version of calculus around 1665.

Today, the ideas of calculus provide us with a way of viewing and analysing the physical world where most things are related to each other. Because they are related, changes in one thing result in corresponding changes in the other things. Because related things are changing, we are often interested in finding the rate at which one thing changes with respect to the other such as the rate of change of distance and velocity with respect to time, the inflation rate of the economy of Papua New Guinea in relation to the way in which the national government controls the money supply, the rate at which medicine works in relation to the doctor's prescription, etc.

Calculus is the main mathematical tool used to analyse and solve physical problems involving rates of change. It also helps us to calculate things like areas, volumes, centre of mass of irregular objects.

In fact calculus is not only a useful tool in mathematics, but it is also useful in other fields of study which include engineering, sciences, economics, astronomy, social sciences, space research, sociology and many more.

There are two main operations in calculus, called **Differentiation** and **Integration**. In this unit we will study the basic ideas of calculus and then look at differentiation. We shall revisit slopes and equations of straight lines and then conclude the unit with simple applications of differentiation and integration.



Learning Outcomes

On successful completion of this module, students will be able to:

- define and explain the concepts of functions
- classify and evaluate function
- sketch and interpret the graphs of functions
- find the average gradient of a curve
- apply rules for differentiation
- calculate tangents and normal of a curve
- state the nature of stationary points and sketching the curves of polynomials
- solve problems on maxima and minima
- find the rate of change using practical problems
- calculate anti-derivatives and indefinite integral
- identify the two theorems on integration
- calculate definite integral
- find area under and between curves



Time Frame

This unit should be completed within 10 weeks.

If you set an average of 3 hours per day, you should be able to complete the unit comfortably by the end of the assigned week.

Try to do all the learning activities and compare your answers with the ones provided at the end of the unit. If you do not get a particular exercise right in the first attempt, you should not get discouraged but instead, go back and attempt it again. If you still do not get it right after several attempts then you should seek help from your friend or even your tutor. Do not pass any question without solving it first.



12.3.1 LIMITS AND CONTINUITY

In the study of calculus, the first important concept or idea that must be introduced is the concept of limit. The idea of limits is fundamental to the study of calculus. In everyday usage, we talk about limiting our calorie intake, speed limits in the highway, or stretching our capacity to the limit. In each of these situations, the use of the word **limit** suggest a boundary that may be reached or not, or possibility exceeded. A mathematical limit has characteristics similar to those of a physical limit. It is the analysis of how function values or outputs change, when inputs change.

In this section, we will investigate the manner in which some functions vary, and whether they approach specific values under certain conditions.

12.3.1.1 Functions

The limit of a function is the cornerstone of both differential and integral Calculus. It is one of the fundamental ideas that distinguish calculus from other areas of mathematics such as algebra and trigonometry. The definition of derivative depends on the notion of the limit of a function.

Let us look at the meaning of a **function**.

A **function, f** , is a special type of a relation such that no two ordered pairs of the set have different second coordinates corresponding to the same first coordinate.

- The set of all the first coordinates (x) of the ordered pairs is the **domain** of the function.
- The set of all the second coordinates (y) of the ordered pairs is the **range** of the function.

If we have a rule or formula giving y in terms of x and there is no more than one value of y for each value of x , then y is said to be a function of x . This can be written in symbols, $y = f(x)$ is read as “ **y is a function of x** ” where **y** is called the **value of the function**.

For example, if $y = f(x) = 5x - 7$, then the value of y depends upon the value of x . for this reason, **y** is called the **dependent variable** and **x** is the **independent variable**. We also say that **y** is expressed explicitly in terms of **x** .

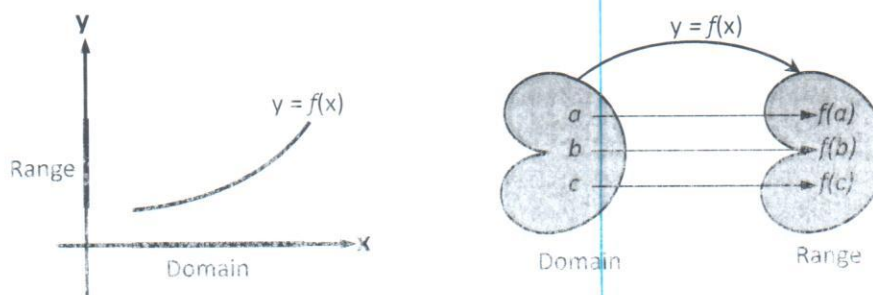
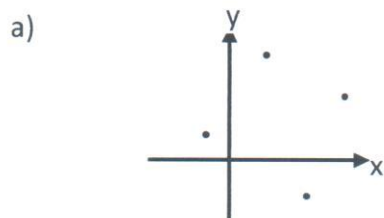


Figure 1: Representation of Domain and Range

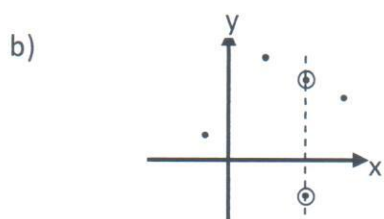
**Notes:**

To be a function means no two ordered pairs of the set can have the same first coordinate and different second coordinates. This means that no vertical line can intersect the graph in more than one point.

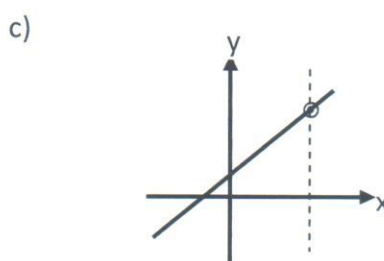
Example 1 Determine from each graph whether it is a function.



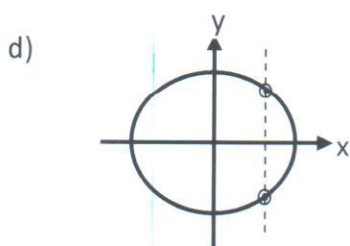
This is a graph of a function because any vertical line meets the graph in no more than one point.



This is not a graph of a function because one vertical line meets the graph in two points.



This is a graph of a function because any vertical line meets the graph in no more than one point.



This is not a graph of a function because the vertical line shown meets the graph in two points.

Evaluating a Function

It is sometimes helpful to enclose x in parentheses wherever it appears in the function formula. If the function is to be evaluated for a particular value of x , then x is replaced by that value within each set of parentheses.

Suppose the function $f(x) = 5x - 7$ is to be evaluated when $x = 2$.

Then $f(x) = 5(x) - 7$ and $f(2) = 5(2) - 7 = 10 - 7 = 3$.

The expression $f(2)$, read “ f of 2” means that x has been replaced by 2. (It does not mean “ f is multiplied by 2.”)



Example 1 Evaluate the function $f(x) = \frac{2x^2 - 7x}{5x - 3}$ when $x = -2, 0, 1$ and h .

Solution: $f(x) = \frac{2x^2 - 7x}{5x - 3}$

a) $f(-2) = \frac{2(-2)^2 - 7(-2)}{5(-2) - 3} = \frac{2(4) + 14}{-10 - 3} = \frac{8 + 14}{-13} = -\frac{22}{13}$

b) $f(0) = \frac{2(0)^2 - 7(0)}{5(0) - 3} = \frac{0 - 0}{0 - 3} = 0$

c) $f(1) = \frac{2(1)^2 - 7(1)}{5(1) - 3} = \frac{2 - 7}{5 - 3} = -\frac{5}{2}$

d) $f(h) = \frac{2(h)^2 - 7(h)}{5(h) - 3} = \frac{2h^2 - 7h}{5h - 3}$

There are different types of function.

1. The Linear Function

The simplest type of function is the **linear function**, where the graph is a straight line.

Any equation of the form $y = mx + b$

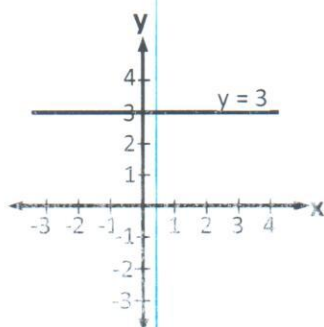
(where **m** is the slope of the line and **b** is the y-intercept) is called a **linear function** in **x**.

Note:

If $m = 0$, then $f(x) = b$, which we called a **constant function**. If the domain of a constant function is a set of real numbers, then the graph of $f(x) = b$ is a **horizontal line**. The line parallel to the y-axis is a **vertical line**. Note that vertical lines are not functions.

Example 1 Sketch the graph of the constant function $f(x) = 3$.

Solution: For any values of x , y is always 3. Hence the graph is a horizontal line 3 unit above the x-axis.



Graph of the constant function $f(x) = 3$.



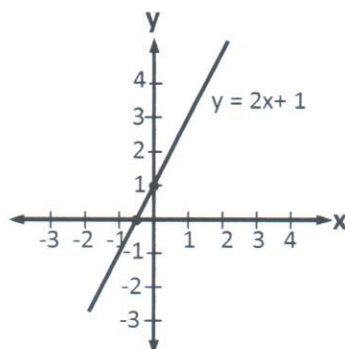
Example 2 Sketch the graph of the linear function $f(x) = 2x + 1$.

Solution:

Since this is a linear function the graph is a straight line.

Make a table of values then plot the points given by the table and join them by line to obtain the graph.

x	0	-0.5	1	2
$f(x) = 2x + 1$	1	0	3	5



Graph of the linear function $f(x) = 2x + 1$

2. The Quadratic Function

Any equation in the form of $y = f(x) = ax^2 + bx + c$ where $a \neq 0$ is called a **quadratic function**. The restriction that $a \neq 0$ is necessary because if $a = 0$, then the equation becomes a linear function. The graph of a quadratic function is a **parabola**.

In a quadratic function $y = f(x) = ax^2 + bx + c$, $a \neq 0$

If $a > 0$, the parabola **opens upward** and has a **minimum value** at the vertex.

If $a < 0$, the parabola **opens downward** and has a **maximum value** at the vertex.

The **vertex** of a parabola is given by the equation

$$x = -\frac{b}{2a} \text{ and } y = \frac{4ac - b^2}{4a}$$

$$\text{Vertex: } \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Example 3

Sketch the graph and determine the vertex, domain and range of $y = x^2$ and $y = -x^2$.

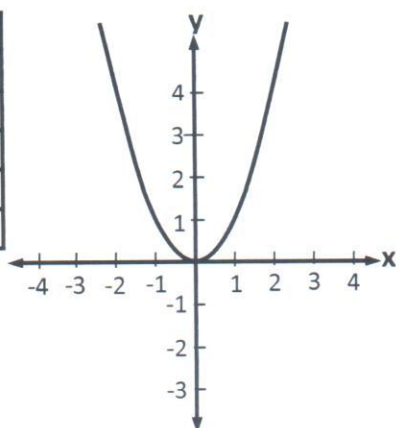
Solution: The vertices of these two parabolas lie at the origin.

The first parabola opens upward (since $a = 1 > 0$).

The second parabola opens downward (since $a = -1 < 0$).

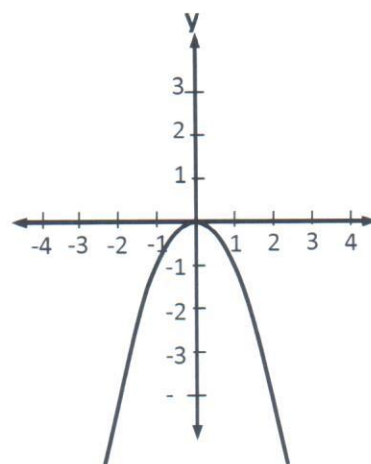


x	$y = x^2$
1	1
2	4
0	0
-1	1
-2	4

The graph of quadratic function $y = x^2$ For $y = x^2$

Domain: $\{x | x \in \mathbb{R}\}$
 Range: $\{y | y \geq 0, y \in \mathbb{R}\}$
 Vertex: $(0, 0)$

x	$y = -x^2$
1	-1
2	-4
0	0
-1	-1
-2	-4

The graph of quadratic function $y = -x^2$ For $y = -x^2$

Domain: $\{x | x \in \mathbb{R}\}$
 Range: $\{y | y \leq 0, y \in \mathbb{R}\}$
 Vertex: $(0, 0)$

3. The Cubic Function

Any equation in the form of $y = f(x) = ax^3 + bx^2 + cx + a$, $a \neq 0$

is called a **cubic function** in x .

Notice that the graph of a cubic function can be traced accurately using the Derivative Test. But, we will illustrate this function when both **b** and **c** are zeros.

Example 4 Determine the domain and range and sketch the graph of $y = f(x) = x^3$.

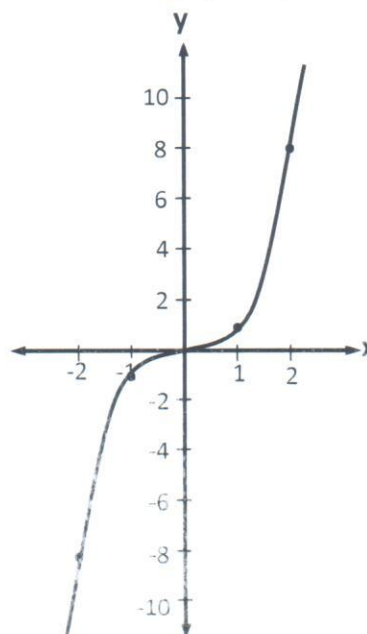
Solution: Table of values for $y = f(x) = x^3$

x	$y = x^3$
1	1
2	8
0	0
-1	-1
-2	-8

For $y = x^3$

Domain: $\{x | x \in \mathbb{R}\}$
 Range: $\{y | y \in \mathbb{R}\}$
 Intercepts: $(0, 0)$

Now do the learning activity.



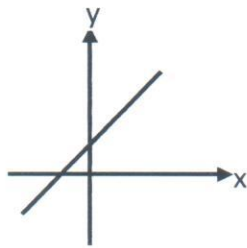


LEARNING ACTIVITY 12.3.1.1

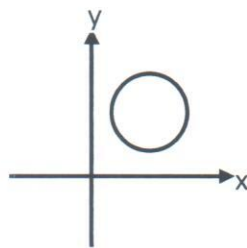


20 minutes

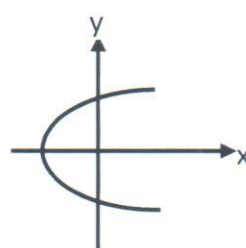
1. Which of the following are graphs of functions?



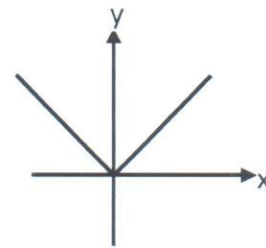
(a)



(b)



(c)



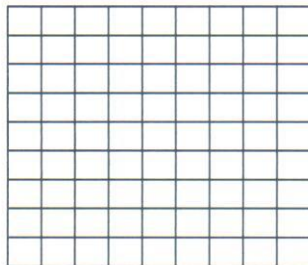
(d)

2. Find the domain and range of the following set of ordered pairs and graph.

a. $\{(2, -1), (3, 4), (0, 2), (-3, -2)\}$

Domain

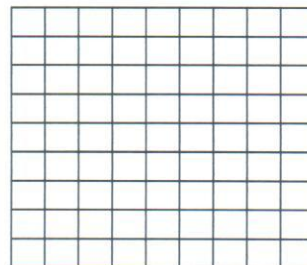
Range:



b. $\{(-4, 0), (0, 0), (3, -2), (1, 5), (-3, -3)\}$

Domain:

Range:



3. Given $f(x) = 3x - 1$, find:

a. $f(2)$

b. $f(0)$

c. $f(x+2)$

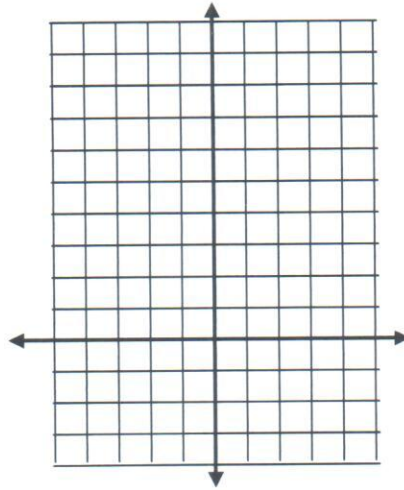
4. Given $f(x) = 2x^2 - 3$, evaluate $\frac{f(5) - f(2)}{6}$.



5. Complete the table of values for the following functions and then draw the graph.

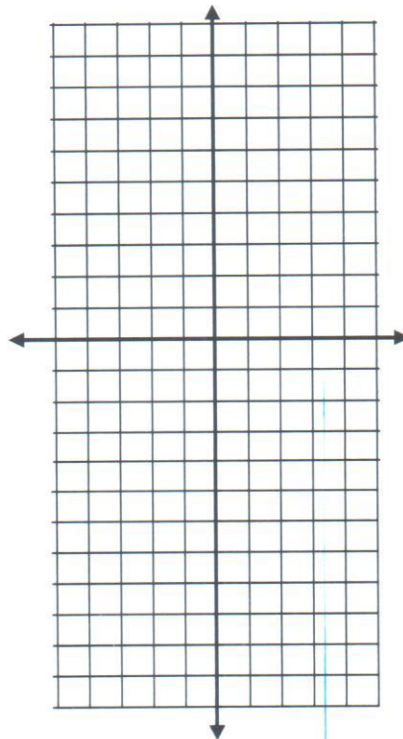
a. $y = f(x) = x^2 - 3x$

x	$y = x^2 - 3x$
3	
2	
1	
0	
-1	
-2	
-3	



b. $y = f(x) = x^3 - 3x$

x	$y = x^3 - 3x$
1	
2	
3	
0	
-1	
-2	
-3	



**12.3.1.2 Limits of a Function**

The limit of a function is the cornerstone of both differential and integral Calculus. It is one of the fundamental ideas that distinguish calculus from other areas of mathematics such as algebra and trigonometry. The definition of derivative depends on the notion of the limit of a function.

Let a function f be defined throughout an open interval containing a real number a except possibly at a itself. We are often interested in the function value $f(x)$ when x is very close to a , but not necessarily equal to a . As a matter of fact, in many instances the number a is not the domain of f , that is, $f(a)$ is undefined. We may ask the question: As x get closer and closer to a but ($x \neq a$), does $f(x)$ get closer and closer to some number L ? If the answer is yes, then we illustrate this concept in the following examples.

Example 1 Evaluate: $\lim_{x \rightarrow 2} (x + 3)$

Solution: We will study the behaviour of linear function $f(x) = x + 3$ as x takes on values closer and closer to 2 from both sides, thus, values less than 2 and values greater than 2.

The table below shows the corresponding values of $f(x) = x + 3$.

x	-2	-1	0	1.99	1.999	2	2.001	2.01	3
$y = f(x)$	1	2	3	4.99	4.999	5	5.001	5.01	6

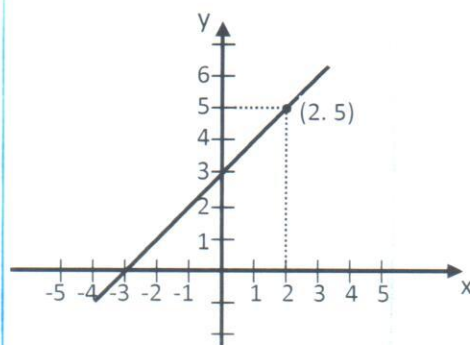
From the table, it follows that the values of $f(x) = x + 3$ come closer and closer to the value 5 as the values of x gets nearer and nearer to 2. In fact, we can find a value of $f(x)$ as close as we please to 5 by choosing a value for x that is close enough to 2. In this case we say that the limit of $f(x)$ as x approaches 2 is 5. Symbolically, we write

$$\lim_{x \rightarrow 2} (x + 3) = 2 + 3 = 5$$

When we write $x \rightarrow 2$ (x approaches 2), we do not assume that $x = 2$, but that x gets closer and closer to 2 from either side. See the graph below. Thus

$x \rightarrow 2^-$ (x approaches 2 from values less than 2)

$x \rightarrow 2^+$ (x approaches 2 from values greater than 2)



Limit of $f(x) = x + 3$ as x approaches 2.



Now evaluating the limit from both sides, we get

$$\lim_{x \rightarrow 2^-} (x + 3) = 5$$

$$\lim_{x \rightarrow 2^+} (x + 3) = 5$$

and

$$\lim_{x \rightarrow 2} (x + 3) = 5$$

Example 2 Evaluate: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

Solution: Let $f(x) = \frac{x^2 - 1}{x - 1}$. Clearly the function is not defined at $x = 1$, that is $f(x)$ does not exist.

Nevertheless, the function does have a limit as $x \rightarrow 1$. Thus,

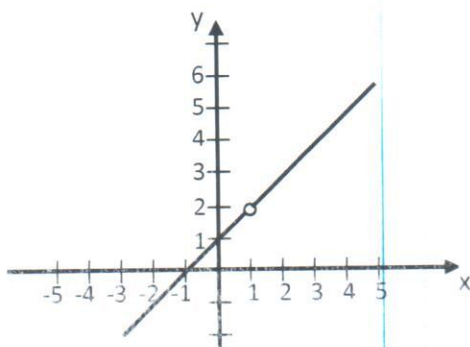
$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{(1)^2 - 1}{1 - 1} = \frac{1 - 1}{0} = \frac{0}{0}, \text{ it is indeterminate.}$$

We can simplify the expression by factoring the numerator. Hence,

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2$$

$$\text{Thus, } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

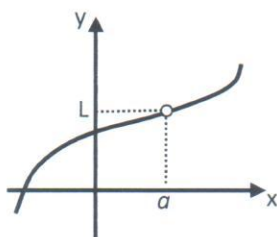
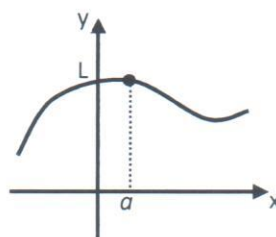
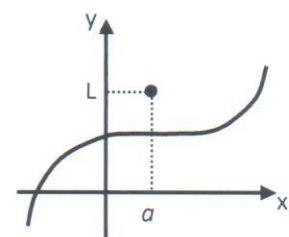
The graph of the function is the line $y = x + 1$, except for a hole at $x = 1$. Thus the function is discontinuous at $x = 1$. Remember that $x \rightarrow 1$ implies that x comes nearer and nearer to 1, but not equal to 1. We are only interested in the behaviour of the function in a small neighbourhood of $x = 1$, the set of all nearby points lying to the left of $x = 1$ and to the right but not in what happens to $f(x)$ at $x = 1$. See graph below.



$$\text{Limit of } f(x) = \frac{x^2 - 1}{x - 1} \text{ as } x \text{ approaches } 1.$$

**Notes:**

1. The notation $\lim_{x \rightarrow a} f(x) = L$, means that as x gets closer to a but not equal to a , $f(x)$ gets closer to L .
2. The limit of a function as x approaches a is independent of the value of the function at a even though $\lim_{x \rightarrow a} f(x)$ exists, the value of the function at a maybe:
 - a. undefined
 - b. the same as the limit
 - c. defined but different from the limit
3. The symbol $\lim_{x \rightarrow a}$ is removed upon evaluation of the limit, or else, it will remain during the simplification process.

a. $f(a)$ is undefinedb. $f(a)$ is the same as the limitc. $f(a)$ is defined but different from the limit

Example 3 Let f be defined by

$$f(x) = \begin{cases} -2 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$$

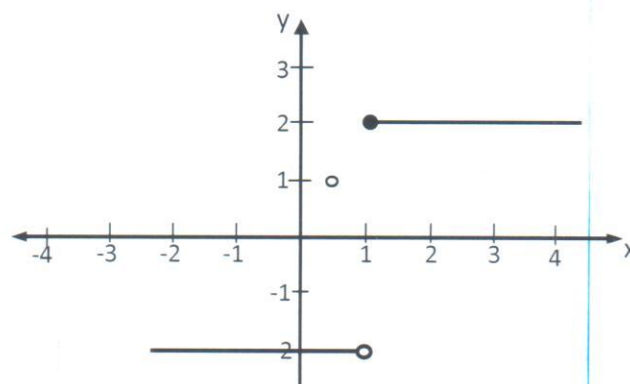
Evaluate:

a. $\lim_{x \rightarrow 1^-} f(x)$

b. $\lim_{x \rightarrow 1^+} f(x)$

c. $\lim_{x \rightarrow 1} f(x)$

Solution: The graph is shown in the figure below.



Now, we evaluate the corresponding limits based on the graph above.

a. $\lim_{x \rightarrow 1^-} f(x) = -2$

b. $\lim_{x \rightarrow 1^+} f(x) = 2$

c. $\lim_{x \rightarrow 1} f(x) = \text{does not exist.}$

Note: $\lim_{x \rightarrow 1} f(x)$ does not exist, because $\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x)$.



Example 4 Evaluate: $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

Solution: Evaluating the limit, we get

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} &= \frac{(4)^2 - 16}{4 - 4} \\ &= \frac{16 - 16}{4 - 4} \\ &= \frac{0}{0}, \text{ it is indeterminate.}\end{aligned}$$

We simplify the expression then evaluate the limit. Hence

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x + 4)(x - 4)}{x - 4} = \lim_{x \rightarrow 4} (x + 4) = 4 + 4 = 8$$

Example 5 Evaluate: $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1}}{\sqrt{x - 1}}$

Solution: Evaluating the limit, we have,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1}}{\sqrt{x - 1}} &= \frac{\sqrt{(1)^2 - 1}}{\sqrt{1 - 1}} \\ &= \frac{\sqrt{1 - 1}}{\sqrt{1 - 1}} \\ &= \frac{0}{0} \quad \text{it is indeterminate.}\end{aligned}$$

Now we express the quotient as a single radical then evaluate the limit.

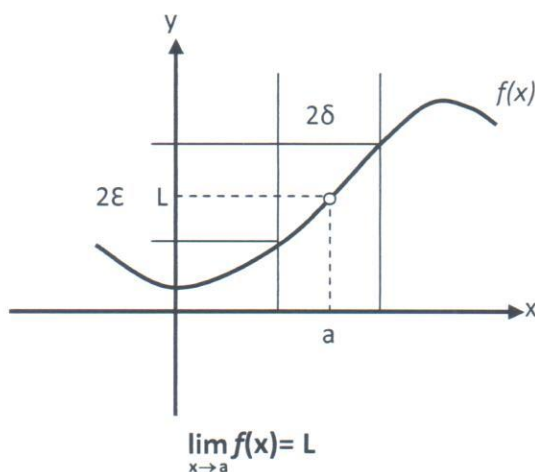
$$\begin{aligned}\text{That is, } \lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1}}{\sqrt{x - 1}} &= \lim_{x \rightarrow 1} \sqrt{\frac{x^2 - 1}{x - 1}} \\ &= \lim_{x \rightarrow 1} \sqrt{\frac{(x + 1)(x - 1)}{x - 1}} \\ &= \lim_{x \rightarrow 1} \sqrt{x + 1} \\ &= \sqrt{1 + 1} \\ &= \sqrt{2}\end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1}}{\sqrt{x - 1}} = \sqrt{2}$$



The Limit of a Function

The **limit of a function** $f(x)$ as x approaches a is L denoted by $\lim_{x \rightarrow a} f(x) = L$ if and only if for any positive number ϵ (no matter how small) there exists a positive number δ (dependent on ϵ) such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. (See Figure below).



This means that if the statement $\lim_{x \rightarrow a} f(x) = L$ is true, then we can find a δ such that $f(x)$ can be made as close to L as we please (i.e. within ϵ of L). By substituting any value of x within a distance δ of a into $f(x)$.

Now do the learning activity.

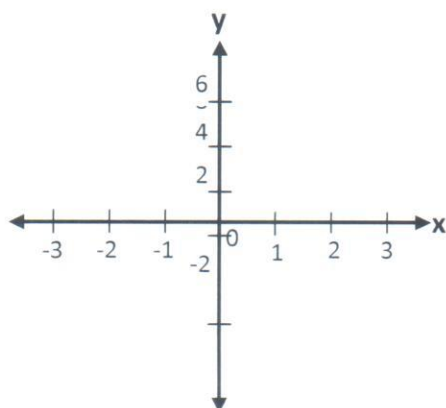


LEARNING ACTIVITY 12.3.1.2



20 minutes

1. Sketch the graph of $f(x) = x^2 - 1$ and use the graph to find $\lim_{x \rightarrow 2} f(x)$.





2. Evaluate the following limit:

a. $\lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x - 1}$

Answer: _____

b. $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 - 2x - 15}$

Answer: _____

c. $\lim_{x \rightarrow -3} \frac{x^2 - 4}{x^2 - 8x + 12}$

Answer: _____

d. $\lim_{t \rightarrow -2} x^2 + 2x + 4$

Answer: _____

- e. A population of flies is growing in a large container. The number of flies P (in hundreds) after t weeks is given by $P = 12t^2 - t^4 + 5$.

Evaluate $\lim_{t \rightarrow 8} P$.

Answer: _____

**12.3.1.3 Properties of Limits**

Let a , n and c be any real numbers.

1. $\lim_{x \rightarrow a} c = c$ where c is a constant. (The limit of a constant is the constant itself).

Example 1

Evaluate the following limits.

a. $\lim_{x \rightarrow a} 2 = 2$

b. $\lim_{x \rightarrow a} 5 = 5$

c. $\lim_{x \rightarrow a} (-6) = -6$

2. $\lim_{x \rightarrow a} x = a$

Example 2

Evaluate the following limits: a. $\lim_{x \rightarrow 5} x$ b. $\lim_{x \rightarrow 0} x$ c. $\lim_{x \rightarrow \frac{2}{3}} x =$

Solution:

a. $\lim_{x \rightarrow 5} x = 5$

b. $\lim_{x \rightarrow 0} x = 0$

c. $\lim_{x \rightarrow \frac{2}{3}} x = \frac{2}{3}$

3. $\lim_{x \rightarrow a} (nx + c) = an + c$

Example 3

Evaluate the following limits: a. $\lim_{x \rightarrow 5} (2x - 1)$ b. $\lim_{x \rightarrow -1} (2 - 5x)$

Solution: a. $\lim_{x \rightarrow 5} (2x - 1) = 2(5) - 1 = 10 - 1 = 9$

b. $\lim_{x \rightarrow -1} (2 - 5x) = 2 - 5(-1) = 2 + 5 = 7$

4. $\lim_{x \rightarrow a} x^n = a^n$, provided a^n exists.

Example 4

Evaluate the following limits: a. $\lim_{x \rightarrow 5} x^3$ b. $\lim_{x \rightarrow \frac{1}{2}} 4x^3$

Solution: a. $\lim_{x \rightarrow 5} x^3 = (5)^3 = 125$

b. $\lim_{x \rightarrow \frac{1}{2}} 4x^3 = 4\left(\frac{1}{2}\right)^3 = 4\left(\frac{1}{8}\right) = \frac{1}{2}$



5. $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$

Example 5

Evaluate the following limits: a. $\lim_{x \rightarrow 8} \sqrt{x}$ b. $\lim_{x \rightarrow 1} \sqrt{x^3 + 2}$

Solution:

a. $\lim_{x \rightarrow 8} \sqrt{x} = \lim_{x \rightarrow 8} \sqrt{8} = 2\sqrt{2}$

b. $\lim_{x \rightarrow 1} \sqrt{x^3 + 2} = \sqrt{(1)^3 + 2} = \sqrt{1 + 2} = \sqrt{3}$

Let $f(x)$ and $g(x)$ be any two functions such that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

6. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

Example 6

Evaluate the following limits.

a. $\lim_{x \rightarrow 2} (2x^3 - x^2 + 5)$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 2} [(2x + 3) \cdot (x^2 + 1)] &= \lim_{x \rightarrow 2} 2x^3 - \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 5 \\ &= 2(2)^3 - (2)^2 + 5 \\ &= 16 - 4 + 5 \\ &= 17 \end{aligned}$$

b. $\lim_{x \rightarrow \frac{1}{2}} \sqrt{4x^2 + 8} - 5x^2$

Solution:

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}} \sqrt{4x^2 + 8} - 5x^2 &= \sqrt{4\left(\frac{1}{2}\right)^2 + 8} - 5\left(\frac{1}{2}\right)^2 \\ &= \sqrt{1 + 8} - \frac{5}{4} \\ &= \sqrt{9} - \frac{5}{4} \\ &= 3 - \frac{5}{4} \\ &= \frac{7}{4} \end{aligned}$$



7. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

Example 7

Evaluate the following limits.

a. $\lim_{x \rightarrow 3} [(2x + 3) \cdot (x^2 + 1)]$

Solution: $\lim_{x \rightarrow 3} [(2x + 3) \cdot (x^2 + 1)] = \lim_{x \rightarrow 3} (2x + 3) \cdot \lim_{x \rightarrow 3} (x^2 + 1)$

$$= [2(3) + 3][(3)^2 + 1]$$

$$= (6 + 3)(9 + 1)$$

$$= (9)(10)$$

$$= 90$$

b. $\lim_{x \rightarrow 4} [(x + 1)(x - 1)] = \lim_{x \rightarrow 4} (x^2 - 1)$

$$= \lim_{x \rightarrow 4} x^2 - \lim_{x \rightarrow 4} 1$$

$$= 4^2 - 1$$

$$= 16 - 1$$

$$= 15$$

8. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$

Example 8

Evaluate: $\lim_{x \rightarrow 2} \frac{x^2 - 2x + 1}{x^2 + 1}$

Solution: $\lim_{x \rightarrow 2} \frac{x^2 - 2x + 1}{x^2 + 1} = \frac{\lim_{x \rightarrow 2} (x^2 - 2x + 1)}{\lim_{x \rightarrow 2} (x^2 + 1)}$

$$= \frac{(2)^2 - 2(2) + 1}{(2)^2 + 1}$$

$$= \frac{4 - 4 + 1}{4 + 1}$$

$$= \frac{1}{5}$$



LEARNING ACTIVITY 12.3.1.3



20 minutes

Evaluate the limit by using the properties of limits.

1. $\lim_{x \rightarrow 5} x$

Answer: _____

2. $\lim_{x \rightarrow -3} 4$

Answer: _____

3. $\lim_{x \rightarrow 1} (2x^3 - 4x + x - 3)$

Answer: _____

4. $\lim_{x \rightarrow -2} (x^2 - 4x + 5)^2$

Answer: _____

5. $\lim_{x \rightarrow \frac{1}{2}} (8x^3 - 2x^2 - 1)$

Answer: _____

6. $\lim_{x \rightarrow 2} \frac{x^2 + x - 2}{x - 1}$

Answer: _____

7. $\lim_{x \rightarrow a} (3x^2 + 4x - 5)$

Answer: _____

8. $\lim_{x \rightarrow -2} \sqrt{x^4 - 4x + 1}$

Answer: _____

**2.3.1.4: Limit of a Quotient**

In this section, we will look at some tools in evaluating the limit of a quotient $\frac{f(a)}{g(a)}$ in which $f(a) = 0$ and $g(a) = 0$. The quotient $\frac{f(x)}{g(x)}$ is said to assume the indeterminate form, $\frac{0}{0}$ at $x = a$, and it is undefined at that point. Nevertheless, the limit of the quotient exists. We will illustrate these tools in the proceeding examples.

Example 1

Evaluate the limit.

a. $\lim_{x \rightarrow 0} \frac{(x+a)-a}{x}$

b. $\lim_{x \rightarrow 0} \frac{(x-2)^2 - 4}{x}$

c. $\lim_{x \rightarrow 0} \frac{(x-h)^3 - x^3}{h}$

Solution:

If the denominator consists of a single term and approaches 0, simplification of the expression should be done first before evaluating the limit.

a. $\lim_{x \rightarrow 0} \frac{(x+a)-a}{x} = \frac{(0+a)-a}{0} = \frac{0}{0}$. It is indeterminate.

We can simplify the expression by removing the grouping symbol.

$$\begin{aligned} \text{Hence, } \lim_{x \rightarrow 0} \frac{(x+a)-a}{x} &= \lim_{x \rightarrow 0} \frac{x+a-a}{x} \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= \lim_{x \rightarrow 0} 1 \\ &= 1 \end{aligned}$$

b. $\frac{(x-2)^2 - 4}{x} = \lim_{x \rightarrow 0} \frac{(0-2)^2 - 4}{0} = \frac{(2)^2 - 4}{0} = \frac{4-4}{0} = \frac{0}{0}$. It is indeterminate.

Here we expand $(x-2)^2 = x^2 - 4x + 4$ and proceed as follows:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^2 - 4x + 4 - 4}{x} &= \lim_{x \rightarrow 0} \frac{x^2 - 4}{x} \\ &= \lim_{x \rightarrow 0} \frac{x(x-4)}{x} \\ &= \lim_{x \rightarrow 0} x - 4 \\ &= 0 - 4 \\ &= -4 \end{aligned}$$



c. $\lim_{x \rightarrow 0} \frac{(x-h)^3 - x^3}{h} = \frac{(x+0)^3 - x^3}{0} = \frac{x^3 - x^3}{0} = \frac{0}{0}$, it is indeterminate.

First we expand $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ then simplify the expression.

$$\begin{aligned} \text{Thus, } \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 \\ &= 3x^2 + 3x(0) + (0)^2 \\ &= 3x^2 \end{aligned}$$

Example 2

Evaluate the limit: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

Solution: We factor the numerator or the denominator to simplify the expression.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{(1)^2 - 1}{1 - 1} = \frac{1 - 1}{0} = \frac{0}{0}, \text{ it is indeterminate.}$$

We factor the numerator using the difference of two squares:

$$\begin{aligned} \text{Thus, } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x \rightarrow 1} (x+1) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Example 3

Evaluate the limit: $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

Solution: First we rationalize the numerator then evaluate the limit.

Multiplying the expression by $\frac{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}$, gives



$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1} &= \frac{\sqrt[3]{x} - 1}{x - 1} \cdot \frac{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} \\&= \lim_{x \rightarrow 1} \frac{(x - 1)}{(x - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)} \\&= \lim_{x \rightarrow 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} \\&= \frac{1}{1 + 1 + 1} \\&= \frac{1}{3}\end{aligned}$$

Now do the learning activities.

**LEARNING ACTIVITY 12.3.1.4**

20 minutes

Evaluate the following limit.

1. $\lim_{x \rightarrow 7} \frac{x^2 - 49}{x - 7}$

Answer: _____

2. $\lim_{x \rightarrow 4} \frac{3x^2 - 8x - 16}{2x^2 - 9x + 4}$

Answer: _____

3. $\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

Answer: _____

4. $\lim_{x \rightarrow 1} \frac{\sqrt{t} - 1}{t - 1}$

Answer: _____

**12.3.1.5: One-sided Limits**

In some cases a function may approach either of two different limits depending on whether the variable approaches its limit through values larger or smaller than that limit. In such a case, the limit is not defined or does not exist but the right-hand and left-hand limits exist.

In this section, we will discuss the behaviour of the function as it approaches from either side to a certain value.

The Right-Hand Limit

The right-hand limit L of a function $f(x)$ at a point a is the limit of the function at a as x approaches a from decreasing values of a , (i.e. as x approaches a from the right). We write the right-hand limit by the notation.

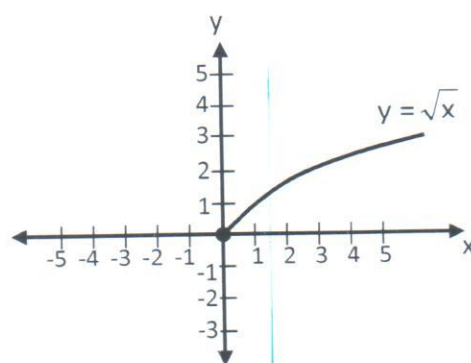
$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow a^+$$

Example 1

Evaluate the limit of $f(x) = \sqrt{x}$ as x approaches 0 from the right.

Solution: The graph of $f(x) = \sqrt{x}$ is shown in Figure below. Notice that the function is defined only for $x \geq 0$. Now evaluating the limit of the function as x approaches 0 from the right, we get

$$\lim_{x \rightarrow 0^+} \sqrt{x} = \sqrt{0} = 0$$



Function with a Right-Hand Limit

The Left-Hand Limit

The left-hand limit L of a function $f(x)$ at a point a is the limit of the function at a as x approaches a from increasing values of a , (i.e. as x approaches a from the left). We write the right-hand limit by the notation.

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow a^-$$



Example 2

Given the split function defined by $f(x) = \begin{cases} 2 & x = 1 \\ -2 & x < 1 \end{cases}$

Evaluate $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$

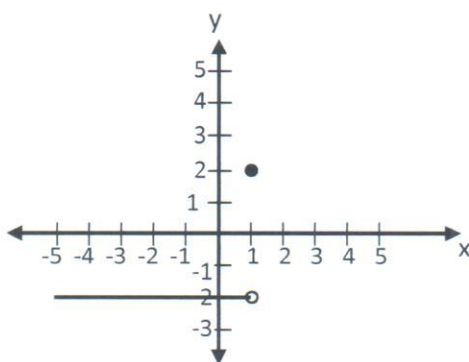
Solution: The graph of $f(x) = \sqrt{x}$ is shown in Figure below. To evaluate the limit as x approaches 1 from the left, we use $f(x) = -2$.

Hence,

$$\lim_{x \rightarrow 1^-} f(x) = -2$$

Notice that $f(x) = 2$ for $x = 1$. Now evaluating the limit as x approaches 1 from the right we get

$$\lim_{x \rightarrow 1^+} f(x) = \text{does not exist}$$



Function with Left-Hand Limit

The Two Sided Limit

The limit exists if and only if the right-hand and the left-hand limits both exist and are equal to L . thus we say that the function has a two-sided limit.

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

Example 3

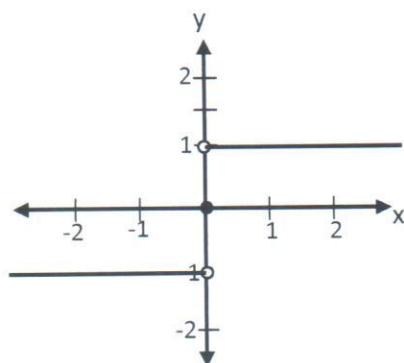
Given the sign function $\text{sgn } x$.

$$\text{sgn } x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Evaluate $\lim_{x \rightarrow 0^-} \text{sgn } x$, $\lim_{x \rightarrow 0^+} \text{sgn } x$ and $\lim_{x \rightarrow 0} \text{sgn } x$.



Solution: The graph is shown below.



Function with a One-Sided Limit

Evaluating the limit, gives

$$\lim_{x \rightarrow 0^-} \operatorname{sgn} x = -1$$

$$\lim_{x \rightarrow 0^+} \operatorname{sgn} x = 1$$

The limit does not exist since the left-hand and the right-hand limits are not equal. Thus,

$$\lim_{x \rightarrow 0} \operatorname{sgn} x = \text{does not exist.}$$

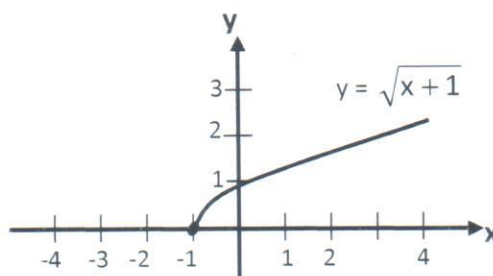
Example 4

Sketch the graph and evaluate the limit of $y = \sqrt{x+1}$ as $x \rightarrow -1$.

Solution:

We sketch the graph of $y = \sqrt{x+1}$ and examine the behaviour of the function as $x \rightarrow -1^-$ and $x \rightarrow -1^+$. (See Figure 2.9)

x	-2	-1	0	1	2	3
y	-	0	1	1.4	1.7	2



Note: the function is not defined for $x < -1$.

We evaluate the limit as follows:

$$\text{i. } \lim_{x \rightarrow -1^-} \sqrt{x+1} = \text{does not exist} \quad \text{ii. } \lim_{x \rightarrow -1^+} \sqrt{x+1} = 0$$

Thus, $\lim_{x \rightarrow -1} \sqrt{x+1} = \text{does not exist}$, since $\lim_{x \rightarrow -1^-} \sqrt{x+1} \neq \lim_{x \rightarrow -1^+} \sqrt{x+1}$.



Example 5

Sketch the graph and evaluate the limit of $f(x) = x, x \neq 2$ as $x \rightarrow 2$.

Solution:

Since the function is not defined at $x = 2$, the graph has a hole on it. (See Figure 2.10)

x	0	1	2	3	4
y	0	1	—	3	4

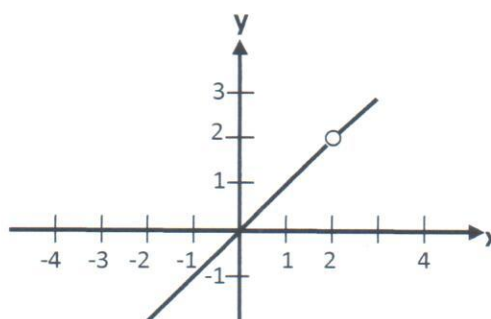


Figure 2.10: Function with a Two-Sided Limit

Evaluating the limit, we obtain

- i. $\lim_{x \rightarrow 2^-} x = 2$
- ii. $\lim_{x \rightarrow 2^+} x = 2$
- iii. $\lim_{x \rightarrow 2} x = 2$



LEARNING ACTIVITY 12.3.1.5



20 minutes

Solve the following problems.

1. Given the $f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$,

Evaluate: (i) $\lim_{x \rightarrow 0^-} f(x)$

Answer: _____

(ii) $\lim_{x \rightarrow 0^+} f(x)$

Answer: _____

(iii) $\lim_{x \rightarrow 0} f(x)$

Answer: _____



2. Given $f(x) = \begin{cases} x - 1, & \text{if } x \leq 3 \\ x^2 - 7, & \text{if } x > 3 \end{cases}$.

Find: (i) $\lim_{x \rightarrow 3^-} f(x)$

Answer: _____

(ii) $\lim_{x \rightarrow 3^+} f(x)$

Answer: _____

(iii) $\lim_{x \rightarrow 3} f(x)$

Answer: _____

**12.3.1.6 Infinite Limits**

In this section, we will discuss another case in which a limit fails to exist. We will discuss functions whose values increase or decrease without bound as the value of x gets closer and closer to a fixed number.

A limit failed to exist if the two one-sided limits are unequal. That is,

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

Example

Evaluate the $\lim_{x \rightarrow 1} \frac{1}{x-1}$, if possible.

Solution: The table below shows the corresponding values of $f(x) = \frac{1}{x-1}$.

x	-2	-1	0	0.5	0.999	1	1.001	1.01	1.1	2	3
y	-.33	-0.5	-1	-2	-1.000	-----	1000	100	10	1	0.5

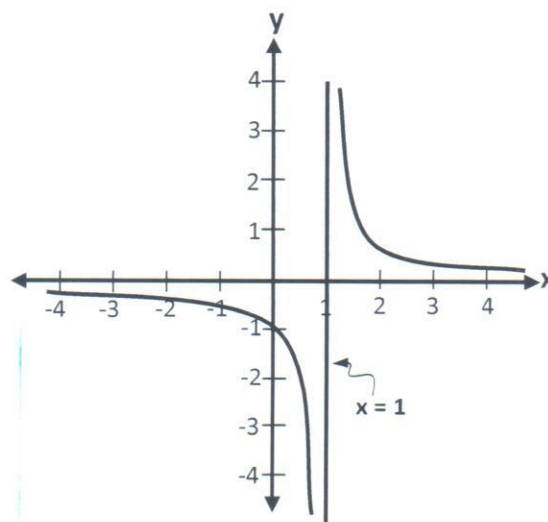


Figure 2.11: Graph of Unbounded Function

We can see in the figure that $f(x)$ decreases without bound as x approaches 1 from the left and $f(x)$ increases without bound as x approaches 1 from the right.

Symbolically, we write

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty$$

Since $f(x)$ is unbounded as x approaches 1, we conclude that the limit does not exist. Hence,

$$\lim_{x \rightarrow 1} \frac{1}{x-1} = \text{does not exist}$$



The **equal sign** in the statement $\lim_{x \rightarrow a} f(x) = \infty$ does not mean that the limit exist. On the contrary, it tells us how the limit fails to exist by denoting the unbounded behaviour of $f(x)$ as it approaches a .

Notice that the symbol ∞ (or $-\infty$) does not denote a number but it is used to describe the situation in which the value of the function is becoming large without bound (or becoming small without bound).

The Limit Approaching Positive Infinity

Let $f(x)$ be a function that is defined at every number in some open interval containing a , except possibly at the number a itself. As x approaches a , $f(x)$ increases without bound, which is written

$$\lim_{x \rightarrow a} f(x) = +\infty$$

if for any number $N > 0$ there exist a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > N$.

The Limit Approaching Negative Infinity

Let $f(x)$ be a function that is defined at every number in some open interval containing a , except possibly at the number a itself. As x approaches a , $f(x)$ decreases without bound, which is written

$$\lim_{x \rightarrow a} f(x) = -\infty$$

if for any number $N > 0$ there exist a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > N$.

Theorem 1: If n is a positive integer, then

a. $\lim_{x \rightarrow 0^+} \frac{1}{x^n} = +\infty$

b. $\lim_{x \rightarrow 0^-} \frac{1}{x^n} = \begin{cases} -\infty, & \text{if } n \text{ is odd} \\ +\infty, & \text{if } n \text{ is even} \end{cases}$

Example

Evaluate the following limits:

a. $\lim_{x \rightarrow 0^+} \frac{1}{x^{12}}$

b. $\lim_{x \rightarrow 0^-} \frac{1}{x^8}$

c. $\lim_{x \rightarrow 0^-} \frac{1}{x^5}$



Solution:

- a. From Theorem 1.a and with $n = 12$, $\lim_{x \rightarrow 0^+} \frac{1}{x^{12}} = +\infty$
- b. From Theorem 1.a and with $n = 8$ which is even, we have $\lim_{x \rightarrow 0^-} \frac{1}{x^8} = +\infty$
- c. From Theorem 1.a and with $n = 5$ which is odd, we have $\lim_{x \rightarrow 0^-} \frac{1}{x^5} = -\infty$

Rules Involving Infinite Limits

Rule 1: Sum Involving Limits

Let a and L be real numbers. If $\lim_{x \rightarrow a} f(x) = L$, then

- i. If $\lim_{x \rightarrow a} g(x) = +\infty$, then $\lim_{x \rightarrow a} [f(x) + g(x)] = L + \infty = +\infty$
- ii. If $\lim_{x \rightarrow a} g(x) = -\infty$, then $\lim_{x \rightarrow a} [f(x) + g(x)] = L - \infty = -\infty$

Example 1

Evaluate the following limits by applying Rule 1.

- a. $\lim_{x \rightarrow 0} (2 + \frac{1}{x^2})$
- b. $\lim_{x \rightarrow 0^-} (\frac{1}{x+1} + \frac{5}{x^3})$

Solutions: Applying Rule 1, we get

- a. $\lim_{x \rightarrow 0} (2 + \frac{1}{x^2}) = \lim_{x \rightarrow 0} 2 + \lim_{x \rightarrow 0} \frac{1}{x^2} = 2 + \infty$
- b. $\lim_{x \rightarrow 0^-} (\frac{1}{x+1} + \frac{5}{x^3}) = \lim_{x \rightarrow 0^-} \frac{1}{x+1} + \lim_{x \rightarrow 0^-} \frac{5}{x^3} = 1 + (-\infty) = -\infty$

Rule 2: Product Involving Infinite Limits

Let a and L be real numbers. If $\lim_{x \rightarrow a} f(x) = L$

- i. If $\lim_{x \rightarrow a} g(x) = +\infty$, then $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \begin{cases} L \cdot \infty = +\infty, & L > 0 \\ -L \cdot \infty = -\infty, & L < 0 \end{cases}$
- ii. If $\lim_{x \rightarrow a} g(x) = -\infty$, then $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \begin{cases} L \cdot -\infty = -\infty, & L > 0 \\ -L \cdot -\infty = +\infty, & L < 0 \end{cases}$

Example 2 Evaluate the following limits by applying Rule 1 and 2.

- a. $\lim_{x \rightarrow 0} (\frac{3}{x+2}) (\frac{4}{x^3})$
- b. $\lim_{x \rightarrow 0^-} (\frac{5-x}{x^3}) (\frac{4-x}{x^2+1})$



Solution: Applying Rules 1 and 2, we get

$$\text{a. } \lim_{x \rightarrow 0} \left(\frac{3}{x+2} \right) \left(\frac{4}{x^3} \right) = \lim_{x \rightarrow 0} \left(\frac{3}{x+2} \right) \cdot \lim_{x \rightarrow 0} \left(\frac{4}{x^3} \right) = \frac{3}{2} (+\infty) = +\infty$$

$$\text{b. } \lim_{x \rightarrow 0^-} \left(\frac{5-x}{x^3} \right) \left(\frac{4-x}{x^2+1} \right) = \lim_{x \rightarrow 0^-} \left(\frac{5-x}{x^3} \right) \cdot \lim_{x \rightarrow 0^-} \left(\frac{4-x}{x^2+1} \right) = (-\infty) \left(\frac{4}{1} \right) = -\infty (4) = -\infty$$

Rule 3: Quotient involving Infinite Limits

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = 0$, then

- i. if $L > 0$ and $g(x) > 0$ for all x close to a , then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{0} = +\infty$
- ii. if $L > 0$ and $g(x) < 0$ for all x close to a , then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{0} = -\infty$
- iii. if $L < 0$ and $g(x) > 0$ for all x close to a , then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{-L}{0} = -\infty$
- iv. if $L < 0$ and $g(x) < 0$ for all x close to a , then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{-L}{0} = +\infty$

Example 3 Evaluate the limit, if it exists.

$$\text{a. } \lim_{x \rightarrow 2^+} \frac{x+1}{x^2+x-6}$$

$$\text{b. } \lim_{x \rightarrow 1^-} \frac{x^2+4}{x-1}$$

$$\text{c. } \lim_{x \rightarrow -5^-} \frac{x-1}{x+5}$$

Solution: Applying Rule 3, we get

$$\text{a. } \lim_{x \rightarrow 2^+} \frac{x+1}{x^2+x-6} = \frac{2+1}{(2)^2+2-6} = \frac{3}{4-4} = \frac{3}{0} = +\infty$$

$$\text{b. } \lim_{x \rightarrow 1^-} \frac{x^2+4}{x-1} = \frac{(1)^2+4}{1-1} = \frac{1+4}{0} = \frac{5}{0} = -\infty$$

$$\text{c. } \lim_{x \rightarrow -5^-} \frac{x-1}{x+5} = \frac{-5-1}{-5+5} = \frac{-6}{0} = +\infty$$



LEARNING ACTIVITY 12.3.1.6



20 minutes

Evaluate the each of the following limits.

1. $\lim_{x \rightarrow 0^+} \frac{3}{x^3}$

Answer: _____

2. $\lim_{x \rightarrow 1^+} \frac{4(x-1)}{x^2-1}$

Answer: _____

3. $\lim_{x \rightarrow 3^-} \frac{x-3}{x^2-6x+9}$

Answer: _____

4. $\lim_{x \rightarrow 0^+} \frac{x^2+3x-3}{x}$

Answer: _____

5. $\lim_{x \rightarrow 3^-} \frac{x^2-9}{x^3-27}$

Answer: _____

**12.3.1.7: Limits at Infinity**

In the previous section, we have discussed infinite limits where the values of the function increased or decreased without bound as the independent variable approaches a real number.

In this section, we will discuss the limit of functions when the independent variable either increases or decreases without bound.

To illustrate this, let us consider the function $f(x) = \frac{1}{x^2}$.

Let x increase without bound ($x \rightarrow +\infty$) and let x decrease without bound ($x \rightarrow -\infty$).

Refer to the Figure 2.12

x	-1	-2	-5	-10	-100	-1000
y	1	0.25	0.04	0.01	0.0001	0.000001

x	1	2	5	10	100	1000
y	1	0.25	0.04	0.01	0.0001	0.000001

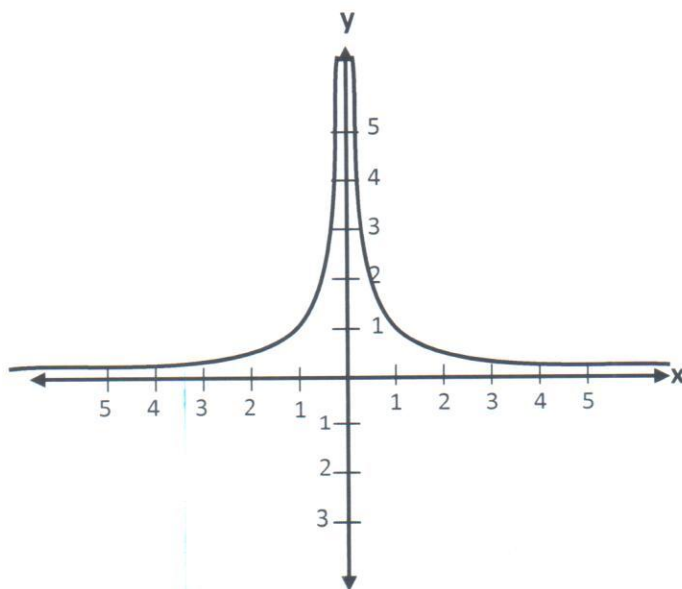


Figure 2.11: Graph of Function where x Increases or Decreases without Bound

We get the following observations from the two tables and graph of $f(x) = \frac{1}{x^2}$; as x increases without bound, $f(x)$ approaches 0, and also as x decreases without bound $f(x)$ approaches 0.

Now using the concepts of limit, we write

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$$

**The Limit as x Increases Without Bound**

Let $f(x)$ be a function that is defined at every number in some interval $(a, +\infty)$, the limit of $f(x)$ as x increases without bound is L , written as

$$\lim_{x \rightarrow +\infty} f(x) = L$$

The Limit as x Decreases Without Bound

Let $f(x)$ be a function that is defined at every number in some interval $(-\infty, a)$, the limit of $f(x)$ as x decreases without bound is L , written as

$$\lim_{x \rightarrow -\infty} f(x) = L$$

The Horizontal Asymptote

A line $y = L$ is called a **horizontal asymptote** of the graph of $y = f(x)$ if

$$\lim_{x \rightarrow +\infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L$$

The Horizontal asymptote occur after evaluating the limit as $x \rightarrow \infty$ then $f(x) \rightarrow L$.

Let us consider the graph of $f(x) = \frac{1}{x^2}$ in Figure 2.11 the horizontal asymptote is the line $y = 0$ (i.e. the x-axis).

Theorem 2: If n is a positive integer, then

$$\text{a. } \lim_{x \rightarrow +\infty} \frac{1}{x^2} = 0 \quad \text{b. } \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$$

Remarks:

To apply the theorem, divide each term of the numerator and denominator by the variable of the highest power occurring in either the numerator or denominator.

Example 1

Evaluate the limit: $\lim_{x \rightarrow +\infty} \frac{3x+1}{4x-1}$

Solution: Dividing each term of the numerator and denominator by x before evaluating the limit, we obtain

$$\lim_{x \rightarrow +\infty} \frac{3x+1}{4x-1} = \lim_{x \rightarrow +\infty} \frac{\frac{3x}{x} + \frac{1}{x}}{\frac{4x}{x} - \frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{3 + \frac{1}{x}}{4 - \frac{1}{x}} = \frac{3+0}{4-0} = \frac{3}{4}$$



Example 2

Evaluate the limit: $\lim_{x \rightarrow -\infty} \frac{x-1}{x^2-1}$

Solution: Dividing each term of the numerator and denominator by x^2 , we get

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow -\infty} \frac{\frac{x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} \\&= \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}} \\&= \frac{0-0}{1-0} \\&= 0\end{aligned}$$

Another solution for Example 2 is by first factoring the denominator to simplify the fraction, and then divide the expression by x .

Hence,

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow -\infty} \frac{x-1}{x^2-1} \\&= \lim_{x \rightarrow -\infty} \frac{x-1}{(x+1)(x-1)} \\&= \lim_{x \rightarrow -\infty} \frac{1}{x+1} \\&= \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}}{1+\frac{1}{x}} \\&= \frac{0}{1} \\&= 0\end{aligned}$$



Example 3

Evaluate the limit: $\lim_{x \rightarrow +\infty} \frac{2x^3 + 1}{1 - x^3}$

Solution:
$$\lim_{x \rightarrow +\infty} \frac{2x^3 + 1}{1 - x^3} = \lim_{x \rightarrow +\infty} \frac{\frac{2x^3}{x^3} + \frac{1}{x^3}}{\frac{1}{x^3} - \frac{x^3}{x^3}} = \frac{2 + \frac{1}{x^3}}{\frac{1}{x^3} - 1} = \frac{2 + 0}{0 - 1} = \frac{2}{-1} = -2$$

Example 4

Evaluate the limit: $\lim_{x \rightarrow +\infty} 2 + \frac{3}{x^3}$

Solution: Since the limit has two terms, we evaluate the limit of each term separately by applying the theorem.

$$\begin{aligned} \lim_{x \rightarrow +\infty} 2 + \frac{3}{x^3} &= \lim_{x \rightarrow +\infty} 2 + \lim_{x \rightarrow +\infty} \frac{3}{x^3} \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

Example 5

Evaluate the limit: $\lim_{x \rightarrow -\infty} \left(\frac{1}{x^4} + \frac{3}{x^2} + \frac{4}{x} - 5 \right)$

Solution: Since there are four terms, we evaluate the limits of each term separately.

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left(\frac{1}{x^4} + \frac{3}{x^2} + \frac{4}{x} - 5 \right) &= \lim_{x \rightarrow -\infty} \frac{1}{x^4} + \lim_{x \rightarrow -\infty} \frac{3}{x^2} + \lim_{x \rightarrow -\infty} \frac{4}{x} - \lim_{x \rightarrow -\infty} 5 \\ &= 0 + 0 + 0 - 5 \\ &= -5 \end{aligned}$$

Example 6

Evaluate the limit: $\lim_{x \rightarrow +\infty} \frac{x-1}{\sqrt{x^2+1}}$

Solution: Divide each term of the numerator by x and the denominator is divided by x^2 because it is inside the radical sign ($\sqrt{\quad}$).

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x-1}{\sqrt{x^2+1}} &= \lim_{x \rightarrow +\infty} \frac{\frac{x}{x} - \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}} = \lim_{x \rightarrow +\infty} \frac{1 - \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}}} = \frac{1 - 0}{\sqrt{1 + 0}} = \frac{1}{\sqrt{1}} = \frac{1}{1} = 1 \end{aligned}$$



LEARNING ACTIVITY 12.3.1.7



20 minutes

Evaluate each of the following limits.

1. $\lim_{x \rightarrow +\infty} \frac{1}{x^8}$

Answer: _____

2. $\lim_{x \rightarrow +\infty} \frac{12}{x^{15}}$

Answer: _____

3. $\lim_{x \rightarrow +\infty} \frac{x^2 + 1}{1 - x^2}$

Answer: _____

4. $\lim_{x \rightarrow -\infty} \frac{3x^3 + 2x + 1}{x^3 + 3x + 2}$

Answer: _____

5. $\lim_{x \rightarrow +\infty} 3 + \frac{1}{x} + \frac{2}{x^2}$

Answer: _____

**12.3.1.8 Continuity**

A function is said to be **continuous** at $x = a$ if there is no interruption in the graph of $f(x)$ at a . Its graph is unbroken at a and there is no hole, jump or gap.

❖ Continuity of a function at a Point

A function is said to be **continuous at a point** $x = a$ if the following conditions are satisfied:

- $f(x)$ is defined, that is, exists, at $x = a$.
- $\lim_{x \rightarrow a} f(x)$ exists.
- $\lim_{x \rightarrow a} f(x) = f(a)$.

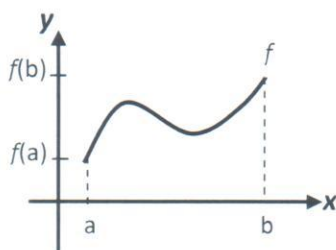
If one or more of these conditions are not satisfied, we say that the function is discontinuous at $x = a$. (See figure below.)



Functions continuous at $x = a$

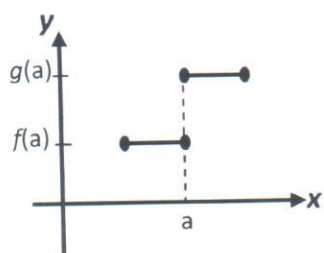
❖ Continuity of a Function on an Open Interval

A function is said to be **continuous on an open interval** (a, b) if it is continuous at each point in that interval. (See figure below.)

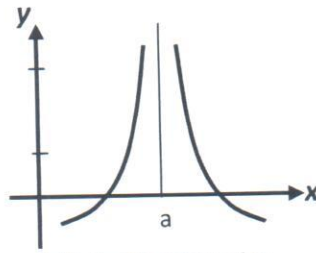


Functions continuous on an interval (a, b)

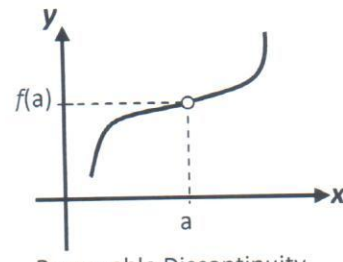
Graphs of functions that are not continuous at a real number a are shown on the next page, where special names are indicated to associate with these discontinuities.



Finite Jump Discontinuity



Infinite Discontinuity



Removable Discontinuity

Graphs of Discontinuous Functions

- A function is said to have a **removable discontinuity** at $x = a$, if

i. $\lim_{x \rightarrow 0} f(x)$ exists, and ii. $\lim_{x \rightarrow 0} f(x) = f(a)$

either because $f(a)$ differs from the value of the limit.

- A function is said to have a **jump discontinuity** at $x = a$, if $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ for any positive integer.

Example 1 Discuss the continuity of $f(x) = 2 - x^3$ at $x = 1$.

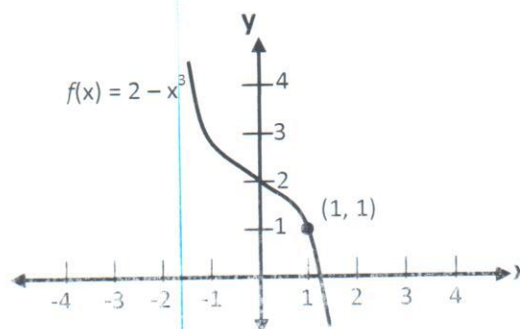
Solution: We apply the definition of continuity to prove that $f(x) = 2 - x^3$ is continuous at $x = 1$. We have,

i. $f(1) = 2 - (1)^3 = 2 - 1 = 1$

ii. $\lim_{x \rightarrow 1} (2 - x^3) = 2 - (1)^3 = 2 - 1 = 1$

iii. $\lim_{x \rightarrow 1} (2 - x^3) = f(1)$

Thus, $f(x) = (2 - x^3)$ is continuous at $x = 1$. See the figure below.

Graph of $f(x) = 2 - x^3$



Example 2 Discuss the continuity of $f(x) = \frac{x^2 - 1}{x - 1}$.

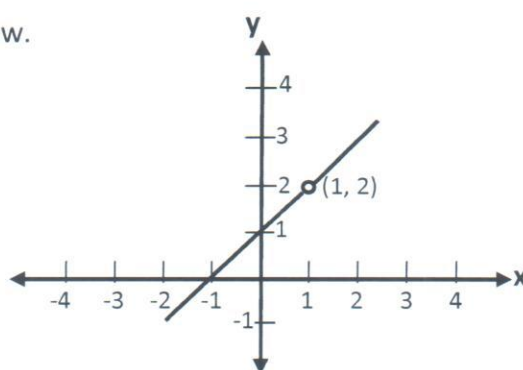
Solution: The function is undefined if the denominator $x - 1$ is zero, that is if $x = 1$. To prove that this function is discontinuous at $x = 1$, using the definition of continuity we get

i. $f(1) = \frac{(1)^2 - 1}{1 - 1} = \frac{1 - 1}{0} = \frac{0}{0}$ it is indeterminate.

ii. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2$

iii. $\lim_{x \rightarrow 1} f(x) \neq f(1)$. Thus, $f(x)$ is discontinuous at $x = 1$.

See figure below.



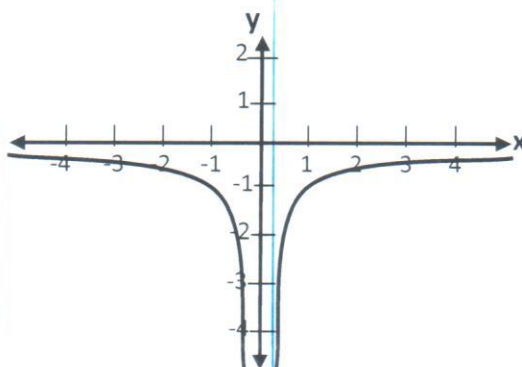
Removable Discontinuity

Example 3 Discuss the continuity of $g(x) = \frac{1}{x^2}$.

Solution: We check the definition of continuity at the point where x approaches 0.

$$\lim_{x \rightarrow 0^-} \left(-\frac{1}{x^2} \right) = -\infty \text{ and } \lim_{x \rightarrow 0^+} \left(-\frac{1}{x^2} \right) = -\infty \Rightarrow \lim_{x \rightarrow 0} g(x) = \infty$$

Since the limit of the function approaches infinity as x approaches 0, we conclude that $g(x)$ has an infinite discontinuity at $x = 0$. (See figure below).



Function with Infinite Discontinuity



Example 4

Discuss the continuity of $f(x) = \begin{cases} 3 & x \leq 1 \\ 2 - x & x > 1 \end{cases}$

Solution: Using the definition of continuity, we get

- When $x = 1$, we use $f(x) = 3$. Thus, $f(1) = 3$.
- Evaluating the right-hand and left-hand limits, respectively, we have

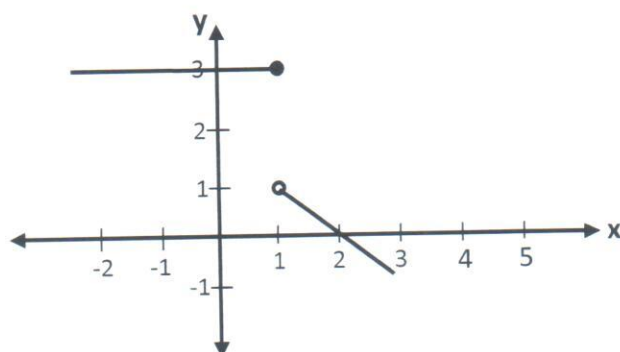
$$\lim_{x \rightarrow 1^+} (2 - x) = 2 - (1) = 1 \quad \text{and} \quad \lim_{x \rightarrow 1^-} 3 = 3$$

$$\lim_{x \rightarrow 1} f(x) = \text{does not exist, since } \lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x).$$

- $\lim_{x \rightarrow 1} f(x) \neq f(1)$.

The graph is discontinuous at $x = 1$ and $f(x)$ has a **finite jump discontinuity**.

(See figure below)



Finite jump discontinuity

Now do the learning activity on the next page.

**LEARNING ACTIVITY 12.3.1.8**

20 minutes

- 1) Determine if the function $f(x) = \begin{cases} 3x - 5 & x \neq 1 \\ 2 & x = 1 \end{cases}$ is continuous at $x = 1$.

2. Determine if the function $f(x) = \frac{x^2 + 1}{x^3 + 1}$ is continuous at $x = -1$.



SUMMATIVE TASK 12.3.1



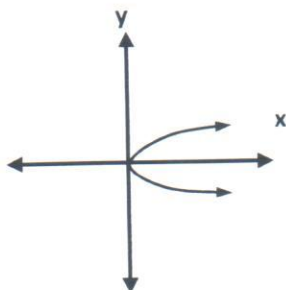
40 minutes

A. Multiple-Choice:

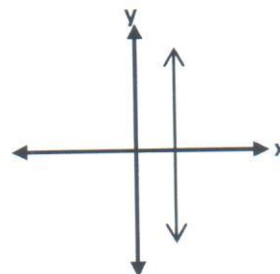
Circle the correct answer for each question from the alternatives A, B, C and D.

1. Which of the following graphs is a function?

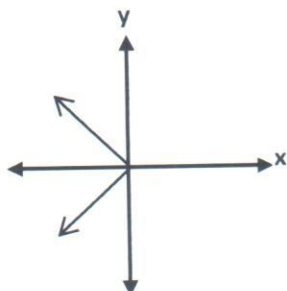
A.



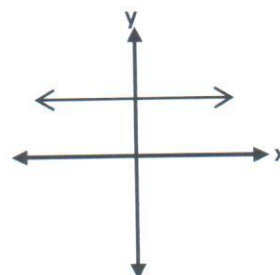
B.



C.



D.

2. Given $f(x) = 3x + 7$, find $f(5)$.

A. 15

B. 22

C. 35

D. 42

3. Evaluate: $\lim_{x \rightarrow -2} \frac{x+8}{x^2-4}$

A. 0

B. -1

C. -2

D. -3

4. Compute $\lim_{x \rightarrow -2} \frac{5x^2 - 8x - 13}{x^2 - 5}$

A. 0

B. 1

C. 2

D. 3



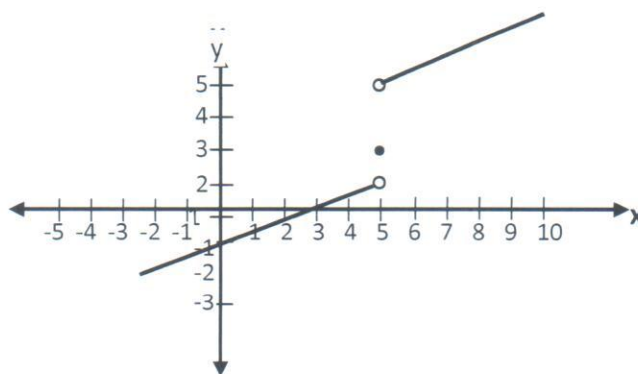
5. Let f be a function defined as $f(x) = \frac{x^2 + 3x - 10}{x^2 - 4}$. Which of the following statements about f is/are true?

- I. $\lim_{x \rightarrow 2} f(x)$ exists
- II. $f(2)$ exists
- III. $f(x)$ is conditional at $x = 2$

- A. I only
B. I and III
C. I and II
D. I, II and III
6. For $x \geq 0$, the line $y = 5$ is an asymptote for the graph of the function of f and $f'(x) \leq 0$. Which of the following statements must be true?

- A. $f(5)$ is undefined
B. $f(0) = 5$
C. $\lim_{x \rightarrow 5} f(x) = \infty$
D. $\lim_{x \rightarrow \infty} f'(x) = 0$

7. The figure below is a graph of $f(x)$.



Which of the following is the value of $\lim_{x \rightarrow 5} f(x)$?

- A. 2
B. 3
C. 5
D. limit does not exist
8. $\lim_{x \rightarrow \infty} \frac{2x^2 + 5x + 2}{x^2 - 16}$ is
- A. 0
B. ∞
C. 2
D. $\frac{5}{4}$



9. Let $f(x) = \begin{cases} x^2 - 1 & \text{if } x \neq 1. \\ 4 & \text{if } x = 1. \end{cases}$

Which of the following statements, I, II and III are true?

I. $\lim_{x \rightarrow 1} f(x)$ exists II. $f(1)$ exists III. f is continuous at $x = 1$

A. I only

B. II only

C. I and II

D. All of them

10. The $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ is equal to

A. 0

B. 1

C. 4

D. does not exist

B. Evaluate each of the following limits.

1. $\lim_{x \rightarrow -2} \frac{-2 - x}{x + 2}$

Answer: _____

2. $\lim_{x \rightarrow 3} \frac{5x^2 - 8x - 13}{x^2 - 5}$

Answer: _____

3. $\lim_{x \rightarrow +\infty} \frac{3x^3 + 2x + 1}{x^3 + 3x + 2}$

Answer: _____



12.3.2: THE DERIVATIVES

In this section, we will see the geometrical interpretation of derivative of a function as the slope of a curve at the point which is just equal to the slope of the tangent line. You will also apply the rules of differentiation to find the derivative of algebraic functions. We also discuss the derivative as a rate of change and solve problems involving rate of change and related rates.

12.3.2.1: The Slope of a Curve

The **tangent line** is defined as the line that intersects the curve at only one point, while the line that intersects the curve in two or more distinct points is called a **secant line**.

The **slope (m)** of the line is defined as the tangent of its angle of inclination θ or equivalently, as the ratio of the change in vertical distance (rise) to the change in horizontal distance (run) as the point moves along the line in either direction. The slope of a line is constant. See Figure 12.3.2.1.

$$m = \tan \theta = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (1)$$

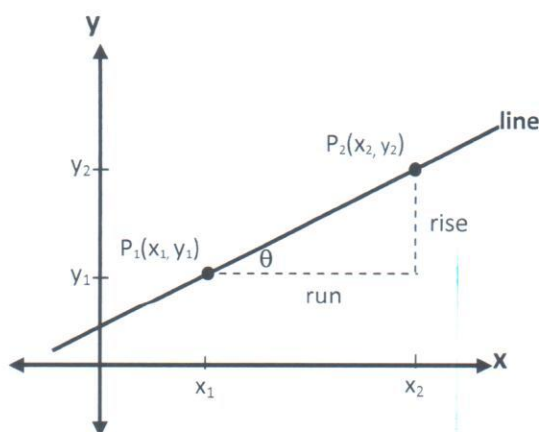


Figure 12.3.2.1: The slope of a line

However, the **slope of a curve is not constant** and must be determined for each particular point of interest. Let the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on the curve. So that we can draw a secant line between the two points on the curve (see Figure 12.3.2.2 at the next page). The slope of the secant line joining the points P_1 and P_2 is given by

$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (2)$$

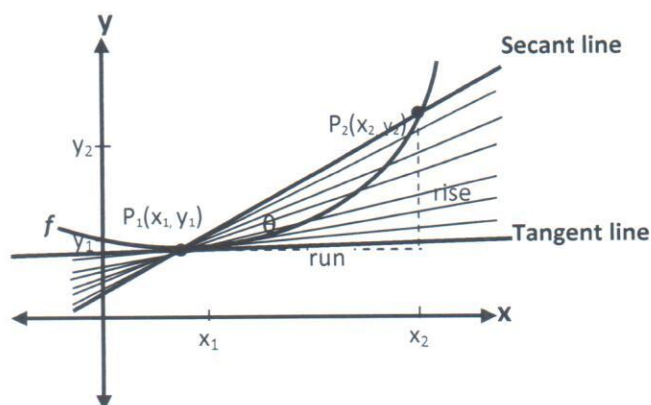


Figure 12.3.2.2: The slope of a curve

Let the point P_1 be fixed while point P_2 be moved along the curve $y = f(x)$ towards the point P_1 . As the point P_2 be moved along the curve $y = f(x)$ towards the point P_1 , the slope of the line joining the points will, in general, vary. That is, as the point P_2 be moved along the curve $y = f(x)$ towards the point P_1 , the slope of the secant line varies by smaller and smaller amounts and approaches a constant limiting value. When this occurs, the limiting value is said to be the slope of a curve at that point. That is, the **first derivative** is interpreted as the **slope of a curve** at that point. Hence,

$$m_{\tan} = \lim_{\Delta x \rightarrow 0} m_{\sec} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{y_2 - y_1}{x_2 - x_1} \quad (3)$$

Let $m_{\tan} = f'(x) = \frac{dy}{dx}$, $\Delta x = x_2 - x_1 \Rightarrow x_2 = x_1 + \Delta x$, also $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Substituting these values into (3), we get

$$m_{\tan} = \lim_{\Delta x \rightarrow 0} m_{\sec} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{y_2 - y_1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_2) - f(x_1)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad (4)$$

In general, for any x , we define the slope of a curve as the first derivative of the function $y = f(x)$ defined by

$$m = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad (5)$$

The Slope of a Curve

The **slope (m) of a curve** $y = f(x)$ at $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$.

Hence,

$$m = \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

provided that the limit exists.



The Normal Line

The **normal line** to a curve $y = f(x)$ at a given point is the line perpendicular to a tangent at that point.

The **slope of the normal line** is equal to the negative reciprocal of the slope of the tangent line.

That is

$$m_n = -\frac{1}{m_t}$$

The Derivative of a Function

The **derivative** of $y = f(x)$ at x is given by

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

provided that the limit exists.

Notes:

- The process for obtaining the derivative of a function is called **differentiation**.
- Various types of notations are used to denote the first derivative of $y = f(x)$ with respect to x , we have

$$f'(x) = \frac{dy}{dx} = D_x y$$

Example 1

Find the derivative of $y = x^2 - 3x + 2$.

Solution: Differentiating y with respect to x , yields

$$\begin{aligned} y = x^2 - 3x + 2 &\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 3(x + \Delta x) + 2] - (x^2 - 3x + 2)}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - (3x - 3\Delta x) + 2 - x^2 + 3x - 2}{\Delta x} \\ \frac{dy}{dx} &= \frac{d}{dx} (x^2 - 3x + 2) = \mathbf{2x - 3} \end{aligned}$$



Example 2

Find the slope and the equation of the tangent and normal lines to the curve $y = x^3$ at $x = 1$.

Solution:

The slope of the tangent line is equal to the slope of the curve at that point.

Thus, finding the derivative of $y = x^3$ at $x = 1$, yields

$$\begin{aligned} m_t = \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \Rightarrow m_t = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ m_t &= \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ m_t &= \frac{dy}{dx} = 3x^2, \text{ at } x = 1 \\ m_t &= 3(1)^2 \\ m_t &= 3(1) = 3 \end{aligned}$$

To find the equation of the tangent line, we use the point-slope form of the equation of a line $y - y_1 = m(x - x_1)$.

To find the point (x_1, y_1) , we substitute $x = 1$ into $y = x^3$.

That is, $y = (1)^3 = 1$. Hence, $(x_1, y_1) = (1, 1)$ and $m = 3$.

Substituting these values into the point-slope form of the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(x - x_1) \Rightarrow y - 1 = 3(x - 1) \\ y - 1 &= 3x - 3 \\ y - 3x &= -2 \end{aligned}$$

Therefore, the equation of the tangent line is **$y - 3x = -2$** .

To find the equation of the normal line, we use the fact that $m_n = -\frac{1}{m_t}$.

Hence $m = -\frac{1}{3}$ (since $m = 3$) and $(x_1, y_1) = (1, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \Rightarrow y - 1 = -\frac{1}{3}(x - 1) \\ 3y - 3 &= -x + 1 \\ 3y + x &= 4 \end{aligned}$$

Therefore, the equation of the normal line is **$3y + x = 4$** .

The graph is shown at the next page.

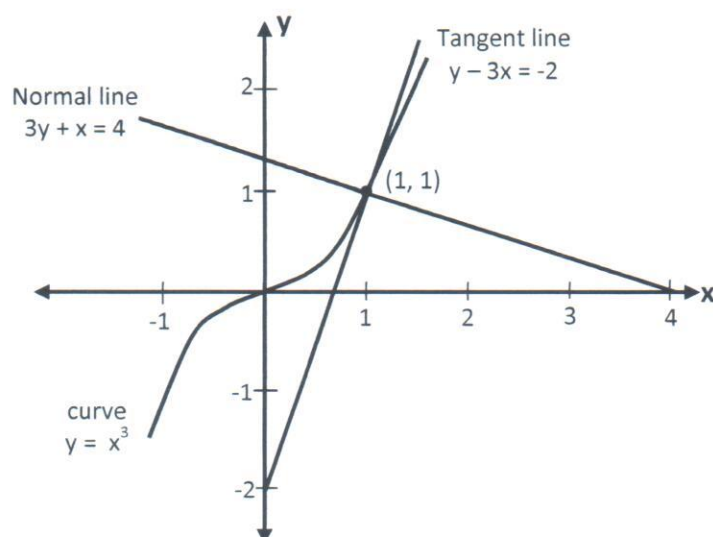


Figure 12.3.2.3: Tangent and Normal Lines

Example 3

Find the equation of the tangent line to a curve $y = x^2 - 1$ that is parallel to the line $2x + y = 6$.

Solution:

The slope of the tangent line to a curve $y = x^2 - 1$ can be computed as follows:

$$\begin{aligned}
 m_1 = \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \Rightarrow m_1 = \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 1] - (x^2 - 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) \\
 &= 2x + 0 \\
 &= 2x
 \end{aligned}$$

Therefore, $m_1 = 2x$

Since the tangent line is parallel to a given line $2x + y = 6$, then it has the same slope as the given line. Reducing the equation of the given line to the slope- intercept form $y = mx + b$, we get

$$2x + y = 6 \Rightarrow y = -2x + 6$$

The slope of the given line is $m = -2$. While the slope of the tangent line is $m_1 = -2$ because the two lines are parallel. Here we can find the point of tangency.

$$2x = -2 \Rightarrow x = -1$$



Substituting $x = -1$ into the given curve, we get the point of tangency.

$$\begin{aligned} f(-1) &= (-1)^2 - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Thus $(-1, 0)$ is the point of tangency (see **Figure 12.3.2.4**)

The equation of the tangent line to the curve $y = x^2 - 1$ at the point $(-1, 0)$ is given by

$$\begin{aligned} y - y_1 &= m(x - x_1) \Rightarrow y - 0 = -2[x - (-1)] \\ y &= -2(x + 1) \end{aligned}$$

Therefore, the equation of the tangent line is given by $y = -2(x + 1)$.

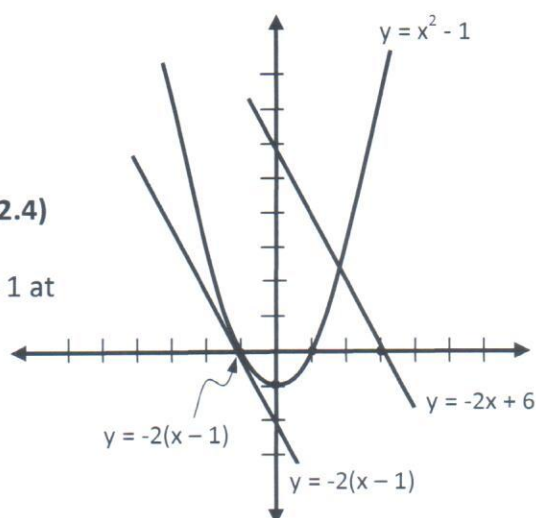


Figure 12.3.2.4

4. Let $f(x) = x^2 - 6x + 2$. Find the point on the graph of f where the tangent line to the graph of f is parallel to the line given by $y = 2x - 1$. Sketch the graph.

Solution:

The slope of the line given by $y = 2x - 1$ is 2.

For tangent line to be parallel to the given line, its slope must be 2.

So we first find x such that $f'(x) = 2$.

$$\begin{aligned} y &= 3x^2 + 2x + 1 \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 6(x + \Delta x) + 2] - (x^2 - 6x + 2)}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 6x - 6\Delta x + 2 - x^2 + 6x - 2}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 - 6\Delta x}{\Delta x} \\ \frac{dy}{dx} &= \frac{d}{dx} (x^2 - 6x + 2) = 2x - 6 \end{aligned}$$

If $f'(x) = 2$, then $2x - 6 = 2 \Rightarrow 2x = 8 \Rightarrow x = 4$.

Since $f(4) = 4^2 - 6(4) + 2 = -6$, the tangent line to the graph of f is parallel to the line given by $y = 2x - 1$ at point $(4, -6)$. See **Figure 12.3.2.5**

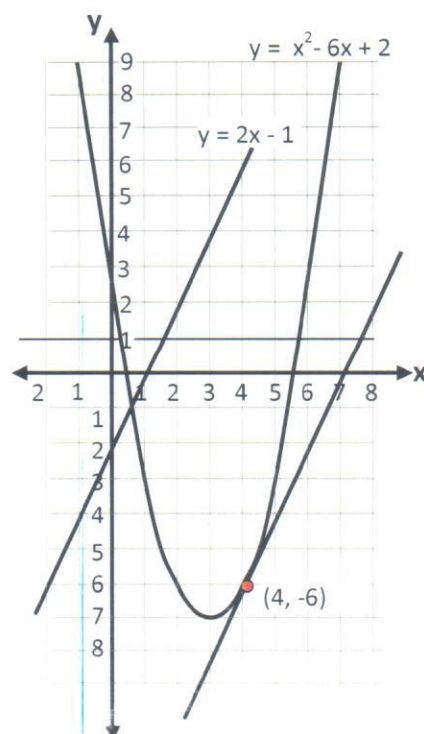


Figure 12.3.2.5



6. Find the slope and the equation of the tangent to the curve $y = 3x^2 + 2x + 1$ at the point $(2, 17)$.

Solution: Since $m \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$$m = f'(2) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{[3(2 + \Delta x)^2 + 2(2 + \Delta x) + 1] - (3(2)^2 + 2(2) + 1)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{12 + 12\Delta x + (3\Delta x)^2 + (4 + 2\Delta x) + 1 - 3x^2 - 17}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{14\Delta x + (3\Delta x)^2}{\Delta x}$$

$$\frac{dy}{dx} = \frac{d}{dx} (3x^2 + 2x + 1) = 14$$

Therefore, the slope (m) is 14.

The equation of the tangent in point-slope form is $y - 17 = 14(x - 2)$.



LEARNING ACTIVITY 12.3.2.1



20 minutes

Solve each of the following:

1. Find the derivative of $y = 3x^2 + 2x + 1$ with respect to x .

Answer: _____

2. Find the equation of the tangent line to the curve of $y = x^2 + 2x + 1$ at the point $(-3, 4)$.

Answer: _____



3. Find the slope of the tangent line to the curve $f(x) = 2x^2 + 3x - 4$ passing through the point $P(-1,5)$.

Answer: _____

4. Find the slope and the equation of the tangent to the curve $y = 3x^2 + 2x + 1$ at the point $(2, 17)$.

Answer: _____

5. Find the equation of the tangent line to the curve $y = x^2$ at the point where $x = 3$.

Answer: _____

**12.3.2.2: Rules for Finding the Derivatives**

In the previous section, we learnt the method of finding the derivative of a function which is tedious (boring) and time consuming. In this section, we will introduce some rules that provide more efficient methods in finding the derivative of a function to avoid lengthy process.

Rule 1: The Derivative of a Constant

Let $f(x)$ be a constant function defined by $y = f(x) = c$, where c is a constant, then $f'(x) = 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

Example 1

Find the derivative of $y = 3$.

Solution: $y = 3$ is a constant function, hence, the derivative is equal to zero.

$$\text{That is, } \frac{dy}{dx} = \frac{d}{dx}(3) = 0$$

Example 2

Find the derivative of $y = 8 + \pi$.

Solution: $y = 8 + \pi$ is a constant function, hence, the derivative is equal to zero.

$$\text{That is, } \frac{dy}{dx} = \frac{d}{dx}(8 + \pi) = 0 + 0 = 0$$

Rule 2: The Derivative of a Power

Let $y = f(x) = x^n$, where n is a positive integer, then for every value of x

$$\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$$

Example 3

Find the derivative of $y = x$.

Solution: Using the Power Rule, we get $\frac{dy}{dx} = \frac{d}{dx}(1)x^{1-1} = x^0 = 1$

**Example 4**

Find the derivative of $y = x^4$.

Solution: Using the Power Rule, we get $\frac{dy}{dx} = \frac{d}{dx}(4)x^{4-1} = 4x^3$

Rule 3: The Derivative of a Constant Times a Function

Let c be a constant. If $f(x)$ is a differentiable function at x , then so is $cf(x)$, and

$$\frac{dy}{dx} = \frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$

In word, a constant factor can be moved through a derivative sign.

Example 5

Find the derivative of $y = 3x^6$.

Solution: $y = 3x^6 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(3x^6) = 3\left(\frac{d}{dx}\right)(x^6) = 3(6)x^{6-1} = 18x^5$

Therefore, $\frac{dy}{dx} = 18x^5$

Example 6

Find the derivative of $y = -\frac{3}{2}x^4$.

Solution: $y = 3x^6 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(-\frac{3}{2}x^4\right) = -\frac{3}{2}\left(\frac{d}{dx}\right)(x^4) = -\frac{3}{2}(4)x^3 = -(3)(2)x^3 = -6x^3$

Therefore, $\frac{dy}{dx} = -6x^3$

Rule 4: The Derivative of Sums and Differences

Let $y = u + v$, where $u = f(x)$ and $v = g(x)$ are differentiable functions of x , then

$$\frac{dy}{dx} = \frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx} = d(u \pm v) = du \pm dv$$



In words, the derivative of a sum of a finite number of differentiable functions is a sum of the derivatives, and the derivative of the difference equals the difference of the derivatives.

Example 7

Find the indicated derivative.

a. $y = 3x^2 + 2x - 1; \frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = 2(3)x + 2(1) - 0$$

$$\frac{dy}{dx} = 6x + 2$$

b. $y = \frac{2}{3}w^3 - 4w + 86; \frac{dy}{dw}$

Solution:

$$\frac{dy}{dw} = 3\left(\frac{2}{3}\right)w^2 - 4(1) + 0$$

$$\frac{dy}{dw} = 2w^2 - 4$$

Rule 5: The Derivative of a Product

Let $y = u \cdot v$, where $u = f(x)$ and $v = g(x)$ are differentiable functions of x , then

$$\frac{dy}{dx} = \frac{d}{dx}[u \cdot v] = u\left(\frac{dv}{dx}\right) + \left(\frac{du}{dx}\right)v = d(u \cdot v) = u dv + v du$$

In words, the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

Example 8

Find the derivative of y with respect to x .

a. $y = (x^3 + 2x)(2x - 1)$

Solution: We differentiate this function using two methods: The Product Rule or first finding the product then differentiate.

Method 1 Using the Product Rule

$$\begin{aligned} y = (x^3 + 2x)(2x - 1) &\Rightarrow \frac{dy}{dx} = (x^3 + 2x)(2) + (2x - 1)(3x^2 + 2) \\ \frac{dy}{dx} &= 2x^3 + 4x + 6x^3 - 3x^2 + 4x - 2 \\ \frac{dy}{dx} &= 8x^3 - 3x^2 + 8x - 2 \end{aligned}$$

**Method 2** Finding the Product

$$y = (x^3 + 2x)(2x - 1) \Rightarrow y = 2x^4 - x^3 + 4x^2 - 2x$$

$$\frac{dy}{dx} = 4(2)x^3 - 3x^2 + 2(4)x - 2$$

$$\frac{dy}{dx} = 8x^3 - 3x^2 + 8x - 2$$

Rule 6: The Derivative of a Quotient

Let $y = \frac{u}{v}$, where $u = f(x)$ and $v = g(x)$ are differentiable functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2} \quad \text{or} \quad d \left(\frac{u}{v} \right) = \frac{vdu - u dv}{v^2}$$

In words, the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Example 9

Find the derivative of $y = \frac{x^2 + 1}{x^3 - 1}$.

Solution: Differentiating using the Quotient Rule: $d \left(\frac{u}{v} \right) = \frac{vdu - u dv}{v^2}$

$$y = \frac{x^2 + 1}{x^3 - 1} \Rightarrow \text{Let } u = x^2 + 1 \text{ so that } \frac{du}{dx} = 2x \text{ and } v = x^3 - 1 \text{ so that } \frac{dv}{dx} = 3x^2.$$

$$\text{Then, } \frac{d}{dx} \left(\frac{x^2 + 1}{x^3 - 1} \right) = \frac{(x^3 - 1)(2x) - (x^2 + 1)(3x^2)}{(x^3 - 1)^2}$$

$$= \frac{2x^4 - 2x - 3x^4 + 3x^2}{(x^3 - 1)^2}$$

$$= \frac{-x^4 - 2x + 3x^2}{(x^3 - 1)^2}$$

$$\frac{dy}{dx} = \frac{x(x^3 + 3x^2 - 2x - 1)}{(x^3 - 1)^2}$$

$$\text{Therefore, } \frac{dy}{dx} \left(\frac{x^2 + 1}{x^3 - 1} \right) = \frac{x(x^3 + 3x^2 - 2x - 1)}{(x^3 - 1)^2}$$



Example 10

Differentiate the function $y = \frac{(x^2 - 1)(x + 1)}{x^2 - 2x + 1}$.

Solution: Using the shortest method, we differentiate the function $y = \frac{(x^2 - 1)(x + 1)}{x^2 - 2x + 1}$ by first simplifying the expression by factoring the numerator and the denominator, then we differentiate the remaining expression.

$$\begin{aligned}\text{So, } y &= \frac{(x^2 - 1)(x + 1)}{x^2 - 2x + 1} = \frac{(x + 1)(x - 1)(x + 1)}{(x - 1)(x - 1)} \\ &= \frac{(x + 1)(x + 1)}{(x - 1)} \\ &= \frac{(x^2 + 2x + 1)}{(x - 1)}\end{aligned}$$

Differentiating the expression, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2 + 2x + 1}{x - 1} \right) \\ &= \frac{(x - 1)(2x + 2) - (x^2 + 2x + 1)(1)}{(x - 1)^2} \\ &= \frac{2x^2 - 2 - x^2 - 2x - 1}{(x - 1)^2} \\ &= \frac{x^2 - 2x - 3}{(x - 1)^2}\end{aligned}$$

$$\text{Hence, } \frac{dy}{dx} \left(\frac{(x^2 - 1)(x + 1)}{x^2 - 2x + 1} \right) = \frac{x^2 - 2x - 3}{(x - 1)^2}.$$

Example 11

Determine the coefficients a , b , and c of the parabola $y = ax^2 + bx + c$ that passes through the point $(3, 13)$ and tangent to the line $8x - y = 15$ at point $(2, 1)$.

Solution: First, we differentiate $y = ax^2 + bx + c$ with respect to x to find the slope of the curve.

Thus,

$$y = ax^2 + bx + c$$

$$\frac{dy}{dx} = 2ax + b \quad (1)$$



Find the slope of the tangent line by solving $8x - y = 15$ for y in terms of x .

Using the slope-intercept formula $y = mx + b$

Hence, $8x - y = 15 \Rightarrow y = 8x - 15$

Therefore, the (slope) $m = 8$ (2)

Since $m = \frac{dy}{dx}$, equating (1) and (2), we obtain

$$2ax + b = 8 \quad (3)$$

At the point of tangency (2, 1), $x = 2$.

Substituting $x = 2$ into (3), we get $4a + b = 8$

$$b = 8 - 4a \quad (4)$$

Since the parabola should pass at the point (3, 13), substituting $x = 3$ and $y = 13$ into the equation $y = ax^2 + bx + c$, we get

$$13 = 9a + 3b + c \quad (5)$$

Substituting (4) into (5), we get

	$13 = 9a + 3(8 - 4a) + c$	
Transposition	$13 = 9a + 24 - 12a + c$	
Collecting like terms	$c = 9a + 12a - 24 + 13$	
	$c = 3a - 11$	(6)

We let $c = 1$ so that $c = 3a - 11$ becomes $a = 4$. Then, substitute this value of a into (4) to get the value of b .

Hence,

$$\begin{aligned} b &= 8 - 4a \\ b &= 8 - 4(4) \\ b &= 8 - 16 \\ b &= -8 \end{aligned}$$

Now, substituting $a = 4$, $b = -8$, and $c = 1$ into (1), we get the equation of the parabola.

$$y = ax^2 + bx + c$$

$$y = 4x^2 - 8x + 1$$

Now do the learning activity on the next page.



LEARNING ACTIVITY 12.3.2.2



20 minutes

Find the indicated derivative.

1. $y = x^5; \frac{dy}{dx}$

Answer: _____

2. $y = x^3 - 2x^2 + 4x - 1; \frac{dy}{dx}$

Answer: _____

3. $y = x^4 - 2x^3 + 3x + 4; \frac{dy}{dx}$

Answer: _____

3. $\frac{w+2}{w-4}; \frac{dy}{dw}$

Answer: _____

5. $y = (x-1)(x+1); \frac{dy}{dx}$

Answer: _____

6. The curves $y = x^2 + bx + c$ and $y = dx - x^2$ have a common tangent line at $(1, 0)$. Find the value of b , c , and d .

Answer: b _____

c _____

d _____

**12.3.2.3: Other Differentiation Rules**

We will continue discussing other rules for finding the derivative of algebraic functions.

Rule 7: The Derivative of a Reciprocal

Let $y = \frac{1}{v}$, where $v = g(x)$ is a differentiable function of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{v} \right) = \frac{\frac{d}{dx}(v)}{v^2} \quad \text{or} \quad d \left(\frac{1}{v} \right) = \frac{dv}{v^2}$$

In words, the derivative of the reciprocal of a function is the negative of the derivative of the function divided by the function squared.

Example 12

Find the indicated derivative. $y = \frac{1}{3t^2 + 4t - 5}$; $\frac{dy}{dt}$

Solution: Using the formula, $d \left(\frac{1}{v} \right) = \frac{dv}{v^2}$, we get

$$\begin{aligned} y = \frac{1}{3t^2 + 4t - 5} &\Rightarrow \frac{dy}{dt} = \frac{d}{dt} \left(\frac{1}{3t^2 + 4t - 5} \right) \\ &= -\frac{6t + 4}{(3t^2 + 4t - 5)^2} \end{aligned}$$

Rule 8: The Derivative of a Function with Rational Exponents

We have already established the power formula for n a positive integer. Now we will extend this formula to rational exponents.

Suppose $n = \frac{1}{m}$ and $y = x^{\frac{1}{m}}$, then

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^{\frac{1}{m}} \right) = \frac{1}{m} x^{\frac{1}{m}-1} \quad \text{or} \quad d \left(x^{\frac{1}{m}} \right) = \frac{1}{m} x^{\frac{1}{m}-1}; \quad m \neq 0$$



Example 13

Find $\frac{dy}{dx}$ if $y = \sqrt{x} + \sqrt[3]{x}$.

Solution: Differentiating y with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} \right) = \frac{1}{2} x^{\frac{1}{2}-1} + \frac{1}{3} x^{\frac{1}{3}-1}$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} \right) = \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$$

Rule 9: The Derivative of a Composite Function and the Chain Rule

We shall establish a formula for finding the derivative of a composite function $y = f[g(x)]$.

Let $y = f(u)$ and $u = g(x)$, so that $y = y = f[g(x)]$, then

$$\frac{dy}{dx} = \frac{d}{dx} \{f[g(x)]\} = f(u) \cdot g(x) = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In words, if y is a function of u and u is a function of x , then, y is a function of x or a composite function. Hence, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x .

Example 14

Find $\frac{dy}{dx}$ if $y = (2x^3 - 4)^3$.

Solution: Let $u = (2x^3 - 4)$ so that $y = u^3$. Then $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 6x^2$

Using the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ substitute these values.

$$\begin{aligned} &= 3u^2(6x^2) \\ &= 3(2x^3 - 4)^2(6x^2) \\ &= 18x^2(2x^3 - 4)^2 \end{aligned}$$



Example 15

Find $\frac{dy}{dx}$ if $y = (2x^3 - 4)^{-2}$.

Solution: Let $u = (2x^3 - 4)$ so that $y = u^{-2}$. Then $\frac{dy}{du} = 2u^{-3}$ and $\frac{du}{dx} = 6x^2$

Using the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ substitute these values.

$$\begin{aligned}
 &= -2u^{-3} (6x^2) \\
 &= -2(2x^3 - 4)^{-3} (6x^2) \\
 &= -12x^2 (2x^3 - 4)^{-3} \\
 &= \frac{-12x^2}{(2x^3 - 4)^3}
 \end{aligned}$$

Rule 10: The General Power Rule

Let $y = [u(x)]^n$, where $u = f(x)$ is a differentiable function of x and n is a real number, then

$$\frac{dy}{dx} = \frac{d}{dx} [u(x)]^n = n[u(x)]^{n-1} \frac{du}{dx} \quad \text{or} \quad d(u^n) = nu^{n-1} du$$

In words, the derivative of the n^{th} power of a differentiable function is the product of n , the $(n - 1)$ st power of the function, and the derivative of the function.

Example 16

Use the general Power Rule to find $\frac{dy}{dt}$ if $y = \sqrt{(t^2 - 2t + 3)^3}$.

Solution: We use the formula: $d(u^n) = nu^{n-1} du$

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{d}{dt} (t^2 - 2t + 3) (t^2 - 2t + 3)^{\frac{3}{2}} \\
 &= \frac{3}{2} (t^2 - 2t + 3)^{\frac{3}{2}-1} (2t - 2) \\
 \frac{dy}{dt} &= 3(t - 1) \sqrt{t^2 - 2t + 3}
 \end{aligned}$$



LEARNING ACTIVITY 12.3.2.3



20 minutes

In problem 1 – 4, find $\frac{dy}{dx}$.

1. $y = (3x^2 - 4x + 5)^4$

Answer: _____

2. $y = (2x^3 - 4x + 1)^{\frac{2}{3}}$

Answer: _____

3. $y = \frac{x^2 - 1}{(x - 1)^3}$

Answer: _____

4. $y = \frac{x}{\sqrt{x+1}}$

Answer: _____

**12.3.2.4: Higher-Order Derivatives**

The derivative of a function is a function itself. In this section, we may consider finding the derivative of the first derivative that is the second derivative and the derivative of the second derivative that is the third derivative, and so on.

Higher-order derivatives are the result of two or more successive differentiation of a function.

For instance, given the function $y = f(x)$ we may define the higher-order derivatives as follows:

$$\frac{dy}{dx} = f'(x) \longrightarrow \text{first derivative}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = f''(x) \longrightarrow \text{second derivative}$$

•
•
•

$$\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right) = f^{(n)}(x) \longrightarrow \text{nth derivative}$$

So long as each successive derivative is differentiable, we can continue in this manner to obtain the derivatives of higher order.

Example 1

Find the first, second and third derivatives of $y = x^5 - 2x^{-3} - 4x + 2$.

Solution: By successive differentiation, we get

$$\frac{dy}{dx} = 5x^4 + 6x^{-4} - 4$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = 20x^3 - 24x^{-5}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = 60x^2 + 120x^{-6}$$



Example 2

Find the first and second derivatives of $y = \frac{1}{\sqrt{x}} + \frac{1}{x}$.

Solution: First, we change the radical into its rational exponent then we differentiate. Hence, the first derivative is found as follows:

$$y = \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x} \quad \Rightarrow \quad y = x^{-\frac{1}{2}} + x^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} - x^{-2}$$

Differentiating again to get the second derivative we have

$$\frac{d^2y}{dx^2} = \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)x^{-\frac{5}{2}} - (-2)x^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4x^{\frac{5}{2}}} + \frac{2}{x^3}$$

$$= \frac{3}{4\sqrt{x^5}} + \frac{2}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4x^2\sqrt{x}} + \frac{2}{x^3}$$

Now do the learning activity.



LEARNING ACTIVITY 12.3.2.4



20 minutes

Find $\frac{d^2y}{dx^2}$, if possible in each of the following:

1. $y = (2x - 3)(3x + 2)$

Answer: _____

2. $y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + 3$

Answer: _____

3. $y = (x - 1)(x^2 + x + 1)$

Answer: _____

4. $y = \frac{3}{2x} + \frac{2}{3x}$

Answer: _____



12.3 2.5: Rate of Change

In the previous section, we discussed how the derivative is used to determine the slope of a curve. In this section we will discuss the derivative as a way of determining the rate of change of one variable with respect to another variable.

In determining the **rate of change** of one variable with respect to another variable, we must be able to distinguish between the **average** and the **instantaneous rates** of change. The two rates of change are comparable to the distinction between the slope of the secant line through two points on a curve and the slope of the tangent line at one point on the curve.

The Average Rate of Change

The **average rate of change** of $y = f(x)$ with respect to x in the interval $(x, x + \Delta x)$ is the slope of the secant line joining the points $[x, f(x)]$ and $[x + \Delta x, f(x + \Delta x)]$ on the graph of $y = f(x)$. See figure 12.3.2.6. It is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{\Delta x}$$

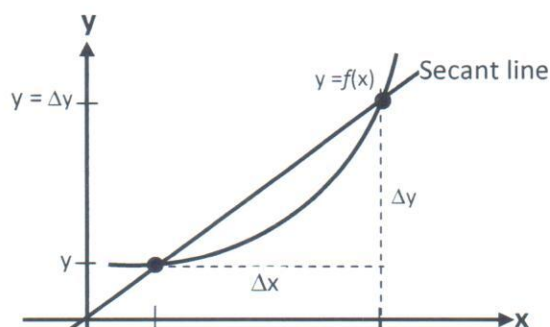


Figure 12.3.2.6: Average rate of Change

The instantaneous Rate of Change

The **instantaneous rate of change** of $y = f(x)$ with respect to x at the point $[x, f(x)]$ is the slope of the tangent line to the graph of $y = f(x)$. See figure 12.3.2.7. It is given by

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

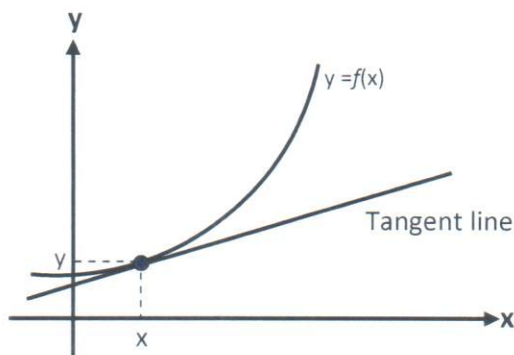


Figure 12.3.2.7: Instantaneous rate of change

Note:

In future work with derivatives, we will use **rate of change** to mean **instantaneous rate of change**.



Example 1

Supposed the function is defined by $y = f(x) = x^2 - x$.

- Find the average rate of change of y with respect to x over the interval $(1, 3)$.
- Find the instantaneous rate of change of y with respect to x at the point $x = -1$.

Solution:

- From the closed interval $(1, 3)$, we get $x_1 = 1$ and $x_2 = 3$. Now using the formula for the average rate of change, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{\Delta x}$$

Notice that $\Delta x = x_2 - x_1 \Rightarrow x_2 = x_1 + \Delta x$.

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{[(3)^2 - 3] - [(1)^2 - 1]}{3 - 1} \Rightarrow \frac{\Delta y}{\Delta x} = 3$$

Thus, on the average y increases 3 units per unit measure on x over the interval $(1, 3)$.

- To find the instantaneous rate of change of y with respect to x , we differentiate the given function with respect to x . That is,

$$y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$$

$$\frac{dy}{dx} = 2(-1) - 1 = -3 \quad \text{at } x = -1$$

Since the instantaneous rate of change is negative at $x = -1$, y is decreasing at the point $x = -1$. See Figure 12.3.2.8

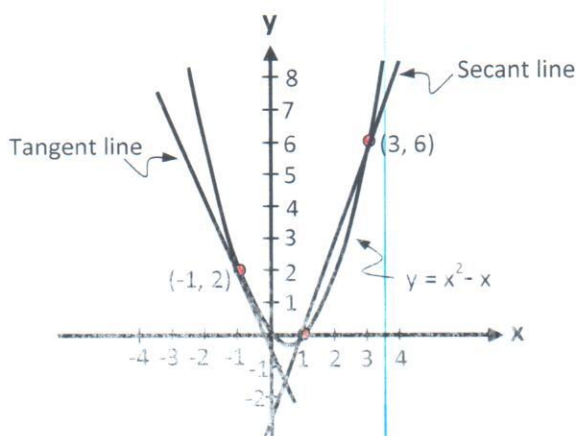


Figure 12.3.2.8: Average and Instantaneous rate of change

Example 2



The formula for finding the area of a circle is given by $A = \pi r^2$. Find:

- the average rate at which the area of a circle changes with r as the radius increases from $r = 1$ to $r = 3$.
- the instantaneous rate of change at which the area of a circle changes with r , when $r = 3$.

Solution:

- We want to find the average rate at which the area of a circle changes with r as the radius increases from $r = 1$ to $r = 3$. Thus,

$$\frac{\Delta A}{\Delta r} = \frac{A_2 - A_1}{\Delta r} = \frac{\pi r_2^2 - \pi r_1^2}{r_2 - r_1} = \frac{\pi(r_2^2 - r_1^2)}{r_2 - r_1} \Rightarrow \frac{\Delta A}{\Delta r} = \frac{\pi(3)^2 - \pi(1)^2}{3 - 1} = 4\pi$$

Therefore, the average rate of increase of the area of the circle with respect to its radius is 4π per unit increase of the radius.

- We find the first derivative of $A = \pi r^2$ with respect to r , when $r = 3$. Hence,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r = 2\pi(3) \Rightarrow \frac{dA}{dr} = 6\pi$$

The Rectilinear Motion

Any motion along a straight line is called **rectilinear motion**.

Velocity and **acceleration** are concepts that drivers of any vehicle knows. As we drive, the speedometer tells us, at any instant, the velocity of our vehicle. As we press down on the gas pedal the velocity of the vehicle changes and we experience acceleration.

In this section, we will show that velocity and acceleration can be calculated by using derivatives.

Let the position of a vehicle **P** on a coordinate line **l** at a time **t** be **s(t)**.

- The **average velocity** Δv of **P** at time t_1 to t_2 is $\Delta v = \frac{\Delta s}{\Delta t}$.
- The instantaneous velocity **v(t)** of **P** at time **t** is $\mathbf{v(t) = s'(t) = \frac{ds}{dt}}$
- The speed of **P** at time **t** is $|\mathbf{v(t)}| = \left| \frac{ds}{dt} \right|$.
- The acceleration **a(t)** of **P** at time **t** is $\mathbf{a(t) = v'(t) = \frac{dv}{dt}}$.

**Note:**

If an object is acted on only by the force of gravity, and if at time $t = 0$ the height of the object is h_0 meters above the ground and the velocity is v_0 meters per second, then the object's height $h(t)$ in meters above the ground at time t seconds is given by $h(t) = -4.9t^2 + v_0t + h_0$.

Example 1

A particle is moving along a horizontal line according to the formula $s = t^2 - 4t + 1$ where s (metre) at any time t is the distance of the particle from the origin at time t (in seconds).

- What is the instantaneous velocity of the particle when $t = 3$?
- When is the particle at rest?
- When is the particle moving to the left?

Solution:

- The instantaneous velocity of the particle at any time is $v(t) = s'(t) = \frac{ds}{dt}$

$$\text{Hence, } s = t^2 - 4t + 1 \quad \Rightarrow \quad v(t) = s'(t) = \frac{ds}{dt} = 2t - 4$$

$$\text{Thus, } v(3) = 2(3) - 4 = 6 - 4 = 2$$

Therefore, the instantaneous velocity of the particle when $t = 3$ is 2 m/sec.

- The particle is at rest when the velocity is 0.

Since its instantaneous velocity at any time t is $v(t) = 2t - 4$, we set $2t - 4$ equals to 0.

$$\begin{aligned} \text{Thus we get, } v(t) &= 2t - 4 \\ &= 0 \end{aligned}$$

Therefore, the particle is at rest 2 seconds since it started moving.

- The particle is moving to the left when its instantaneous velocity is negative. (It is moving to the right when its instantaneous velocity is positive).

$$\text{If } v(t) = 0, \text{ we get } 2t - 4 < 0 \text{ so that } t < 2.$$

Thus, the particle is moving to the left on the time interval $(0, 2)$.



Example 2

Suppose a twenty kina coin is dropped from a height of 100 meters, its height s at time t is $s(t) = -4.9t^2 + 100$ where s is measured in meters and t is measured in seconds. Find:

- The average velocity on the interval $(1, 3)$
- The instantaneous velocity when $t = 2$ seconds
- How long will it take the object to hit the ground?
- The velocity of the object when it hits the ground.

Solution:

- a. To find the average velocity, we use the formula: $\Delta v = \frac{\Delta s}{\Delta t}$ at $t_1 = 1$ and $t_2 = 3$.

$$\begin{aligned}\text{Hence, } \Delta v &= \frac{\Delta s}{\Delta t} = \frac{s(3) - s(1)}{t_2 - t_1} = \frac{(-4.9(3)^2 + 100) - (-4.9(1)^2 + 100)}{3 - 1} \\ &= \frac{-44.9 + 100 + 4.9 - 100}{2} \\ &= \frac{-39.2}{2} \\ &= -19.6\end{aligned}$$

Therefore, the average velocity is -19.6 meter per second.

- b. To find the instantaneous velocity of the coin, we differentiate $s(t) = -4.9t^2 + 100$ with respect to t then substitute $t = 2$ seconds.

$$\begin{aligned}\text{Differentiating } s(t) = -4.9t^2 + 100 &\Rightarrow v(t) = \frac{ds}{dt} = -4.9t^2 + 100 \\ &= -2(4.9)t \\ &= -9.8t\end{aligned}$$

$$\begin{aligned}\text{Substituting } t = 2 \text{ seconds, we get } v(t) &= \frac{ds}{dt} = -9.8t = -9.8(2) \\ &= -19.6 \text{ meter per seconds}\end{aligned}$$

- c. To find the time when the coin hits the ground, we solve for t when $s = 0$.

$$\begin{aligned}s &= -4.9t^2 + 100 \\ 0 &= -4.9t^2 + 100 \\ 4.9t^2 &= 100 \\ t^2 &= \frac{100}{4.9} \\ t &= \sqrt{\frac{100}{4.9}} = 4.52 \text{ sec}\end{aligned}$$



- d. To find the final velocity when the coin hits the ground, we substitute $t = 4.52$ seconds into $v(t) = \frac{ds}{dt} = -9.8t$.

Thus we have,

$$\begin{aligned}v(t) &= \frac{ds}{dt} = -9.8t \\&= -9.8(4.52 \text{ seconds}) \\&= -44.3 \text{ meter per seconds}\end{aligned}$$

Therefore, the velocity of the coin when it hits the ground is **-44.3 mps**.

Now do the learning activity.



LEARNING ACTIVITY 12.3.2.5



20 minutes

Solve the following problems.

1. A particle is moving along a horizontal line according to the formula $s = t^3 - 6t^2 + 9t - 15$ where s (in m) is the distance of the particle from the origin at time t (in seconds).
 - a. What is the instantaneous velocity of the particle when $t = 4$ seconds?

Answer: _____

- b. When is the particle at rest?

Answer: _____

- c. When is the particle moving to the right?

Answer: _____



2. A projectile is fired straight upward. Its distance above the ground after t seconds is given by $s(t) = -4.9t^2 + 300t$.

a. Find the time at which the projectile hits the ground.

Answer: _____

b. Find the velocity at which the projectile hits the ground.

Answer: _____

c. What is the maximum altitude achieved by the projectile?

Answer: _____

d. What is the acceleration at any time?

Answer: _____

3. A stone is dropped from the top of a bridge lands in a river below 10 sec later.

a. What distance does the stone travel from the bridge to the water?

Answer: _____

b. With what speed does the stone hit the water?

Answer: _____

**12.3 2.6: Related Rates**

In this section, we will consider problems involving two (or more) variables that are changing with respect to time t . In such problems we want to find the rate at which some quantity is changing by relating to other quantities whose rates of change are known. If two such quantities are related to each other, then their rates of change with respect to time t are also related.

For instance, if x and y are related by the equation $x^2 + y^2 = 1$, then differentiating this equation with respect to t produce an equation involving the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\text{Hence, } 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \left(\frac{x}{y} \right) \left(\frac{dx}{dt} \right)$$

The derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are called **related rates**, since they are related by means of an equation.

Strategy for solving related rates

1. Draw the figure and label the variables and constants. Use t for time. Assume all variables are differentiable functions of t .
2. Write an equation relating the quantity with unknown rate of change to quantities whose rates of change are known. Do not substitute the known values of the variables until after taking the derivative.
3. Differentiate both sides of this equation with respect to time and solve for the derivative that will give the unknown rate of change.
4. Substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Example 1

When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. at what rate is the plate's area increasing when the radius is 50 cm?

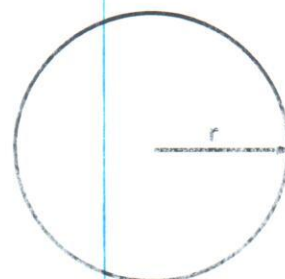
Solution:

Step 1: We draw a picture of the circular plate.

We let the variable t represent time (in min).

A = area (in cm^2) of the circular plate at time t

r = radius (in cm) of the plate at time t .





Step 2: At the time in question, $r = 50$ cm and $\frac{dr}{dt} = 0.01$ cm/min.

Step 3: Find $\frac{dA}{dt}$

Step 4: How are the variables related?

$$A = \pi r^2 \quad (\text{formula for the area of a circle in terms of radius})$$

Step 5: Differentiating both sides of the formula with respect to t (keeping in mind that A and r are both functions of t), we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Step 6: Substituting the values of r and $\frac{dr}{dt}$ into the equation in Step 5.

$$\begin{aligned} \frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\ &= 2\pi(50 \text{ cm})(0.01 \text{ cm/min}) \\ &= \pi \text{ cm}^2/\text{min} \end{aligned}$$

Therefore, the area of the plate is increasing at the rate $\pi \text{ cm}^2/\text{min}$ when the radius is 50 cm.

Example 2

The top of a 10 m ladder long rests on a vertical wall of a residential building while the bottom rests on a horizontal ground. If the top slides down at the rate of 20 m/min, how fast is the lower end moves along the ground when the lower end is 8 meters from the wall?

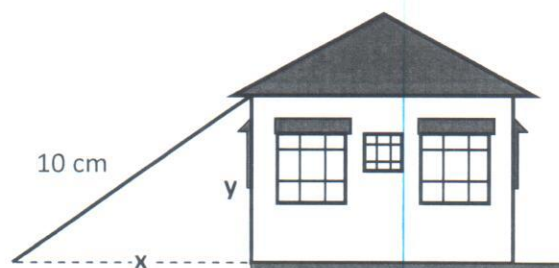
Solution:

Step 1: We draw the figure and label the desired quantities.

Let t = time elapsed since the ladder started to slide down.

y = distance from the ground to the top of the ladder

x = distance from the bottom of the ladder to the wall.



Step 2: We want to find $\frac{dx}{dt}$ when $x = 8$ meters.

Step 3: Using the Pythagorean Theorem we can describe the relationship between x and y .



Hence,

$$(10)^2 = x^2 + y^2 \quad \Rightarrow \quad 100 = x^2 + y^2$$

Step 4: Since x and y are functions of time t , we can differentiate both sides of the equation with respect to t and obtain

$$100 = x^2 + y^2$$

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = -\frac{y}{x} \left(\frac{dy}{dt} \right)$$

Step 5: We can divide, since $x = 8$ m, $y = 6$ m and $\frac{dy}{dt} = -20$ m per min.

Hence,
$$\frac{dx}{dt} = -\frac{6\text{m}}{8\text{m}}(-20\text{m/min}) \quad \Rightarrow \quad \frac{dx}{dt} = 15 \text{ m/min}$$

Therefore, the ladder moves along the ground at the rate of 15 m/min.

Example 3

A cylindrical tank of radius 10 feet is being drained with water at the rate of 200 ft^2 per min. How fast is the height of water in the tank changing?

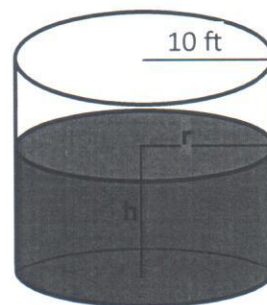
Solution:

Step 1: We draw figure of the cylindrical tank.

Let the variable t represent time (in min).

V = volume (ft^3) of water in the tank at time t .

h = height (ft) of water in the tank at time t



Step 2: At any time, r = radius = 10 ft and $\frac{dV}{dt} = -200 \text{ ft}^2/\text{min}$

($\frac{dV}{dt}$ is negative since the volume of water is decreasing as time increases.)

Step 3: Find $\frac{dh}{dt}$.

Step 4: The variables are related using the formula for the volume of a cylinder.

$$V = \pi r^2 h$$

$$V = \pi(10)^2 h \quad \text{since } r = 10 \text{ from Step 2.}$$

$$V = 100\pi h$$



Step 5; Differentiating both sides of the equation in Step 4 with respect to t (keeping in mind that V and H are function of r), we obtain

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

Solving for $\frac{dh}{dt}$ we have, $\frac{dh}{dt} = \frac{1}{100\pi} \frac{dV}{dt}$

Step 6: Since $\frac{dV}{dt} = -200 \text{ ft}^3/\text{min}$, $\frac{dh}{dt} = \frac{1}{100\pi} (-200) = -\frac{2}{\pi}$.

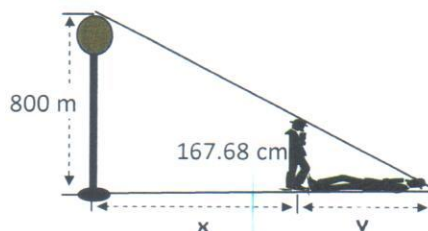
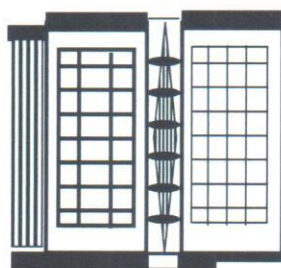
Therefore, the height of the water is decreasing at the rate of $-\frac{2}{\pi} \text{ ft/min}$.

Example 4

A man 167.68 cm tall is walking directly away from a lamp post fronting a building at the rate of 91.46 cm/sec. If the lamp is 800 cm above the ground, find

- the rate at which his shadow is lengthening, and
- the rate at which the tip of his shadow is moving.

Solution: We draw a figure and label the desired quantities.



Let x = length of man's shadow
 y = distance of man from the lamp post

$\frac{ds}{dt}$ = rate at which the man's shadow is lengthening

$\frac{dw}{dt}$ = rate at which the tip of the man's shadow is moving

$\frac{dx}{dt} = 91.46 \text{ cm/sec}$ = rate at which the man is moving away from the lamp post

By similar triangles, we can find the rate at which the man's shadow is lengthening.

$$\frac{s}{167.68} = \frac{x+s}{800} \Rightarrow 800s = 167.68x + 167.68s$$

$$800s - 167.68s = 167.68x$$

$$632.32s = 167.68x$$

$$s = 0.27x$$



Differentiating $s = 0.27x$ with respect to t , we obtain

$$\frac{ds}{dt} = 0.27\left(\frac{dx}{dt}\right) \Rightarrow \frac{ds}{dt} = 0.27(91,46) = 24.96 \text{ cm/sec}$$

Let $w = r + s$. Notice that the tip of the man's shadow is moving in a line, so we differentiate w with respect to t and get

$$w = r + s \Rightarrow \frac{dw}{dt} = \frac{dX}{dt} + \frac{ds}{dt}$$

Substitute $\frac{ds}{dt} = 24.96 \text{ cm/sec}$ and $\frac{dX}{dt} = 91.46 \text{ cm/sec}$ into $\frac{dw}{dt} = \frac{dX}{dt} + \frac{ds}{dt}$ to get the rate at which the tip of the man's shadow is moving

$$\frac{dw}{dt} = \frac{dX}{dt} + \frac{ds}{dt} \Rightarrow 91,46 + 24.96 = \mathbf{116.15 \text{ cm/sec}}$$

Now do the learning activity.



LEARNING ACTIVITY 12.3.2.6



20 minutes

Solve the following problems.

1. A 13 feet ladder is leaning against a wall when its base starts to slide away. By the time the base is 12 feet from the wall, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall then?

Answer: _____

2. The radius of a sphere is increasing at the rate of 2 meters per second. Find the rate of change of the volume with respect to time t when $r = 4$ meters.

Answer: _____



3. A light is 20 meters above a straight horizontal path. If a man 1.65 meters tall is walking away from the light at the rate of 1.8 meters per second, how fast is his shadow lengthening?

Answer: _____

4. A cylindrical tank of radius 8 feet is being drained with water at the rate of $192 \text{ ft}^3/\text{minute}$. How fast is the height of water in the tank changing?

Answer: _____

**12.3.3.7 Derivatives of Trigonometric Functions**

In this section, we will derive the formulas for the derivatives of the trigonometric functions and differentiate functions which involve trigonometric functions.

Let us recall the following fundamental identities.

Fundamental identities

$$\csc \theta \sin \theta = 1$$

$$\sec \theta \cos \theta = 1$$

$$\cot \theta \tan \theta = 1$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

The derivatives of sine and cosine functions were derived from these theorems.

$$1. \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$2. \quad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

Example 1

Find $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$.

Solution: in order to apply Theorem 1, rewrite the function as follows:

$$\frac{\sin 5x}{x} = \frac{5}{5} \left(\frac{\sin 5x}{x} \right) = 5 \frac{\sin 5x}{5x}$$

Notice that as $x \rightarrow 0$, $5x \rightarrow 0$, and so by Theorem 1 with $\theta = 5x$, we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{x} &= \lim_{x \rightarrow 0} 5 \left(\frac{\sin 5x}{5x} \right) \\ &= (5) \left[\lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \right) \right] \\ &= 5(1) \\ &= 5 \end{aligned}$$



Example 2

$$\text{Find } \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}$$

$$\begin{aligned} \text{Solution: } \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} &= \lim_{\theta \rightarrow 0} \left[\left(\frac{\cos \theta - 1}{\sin \theta} \right) \left(\frac{\theta}{\theta} \right) \right] = \lim_{\theta \rightarrow 0} \left[\left(\frac{\cos \theta - 1}{\theta} \right) \left(\frac{\theta}{\sin \theta} \right) \right] \\ &= \left(\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \right) \left(\lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \right) = (0) \left(\frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} \right) = 0(1) = 0 \end{aligned}$$

Derivatives of Trigonometric Functions

The development for the derivative of $\cos x$ is similar to that of $\sin x$. The derivatives of the remaining functions are computed from their definitions in terms of $\sin x$ and $\cos x$, and using the Quotient Rule for derivatives.

We now have the following differentiation formulas:

- The derivatives of $\sin x$

$$\frac{d}{dx}(\sin x) = \cos x$$

- The derivative of $\cos x$

$$\frac{d}{dx}(\cos x) = -\sin x$$

- The derivative of $\tan x$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

- The derivative of $\csc x$

$$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

- The derivative of $\sec x$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

- The derivative of $\cot x$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$



Example 1

Differentiate $\cos^3 x$ with respect to x .Solution: Let $y = \cos^3 x$ Let $u = \cos x$

$$\text{Therefore, } y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = -\sin x$$

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= \frac{dy}{dx} = (-\sin x)(3u^2) \\ &= (-\sin x)(3\cos^2 x) \\ &= -3\cos^2 x \sin x \end{aligned}$$

Example 2

Differentiate $y = 3 \sin x - 4 \cos x$.Solution: Use the formula $\frac{dy}{dx} = \frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$

$$\begin{aligned} \text{We have } y = 3 \sin x - 4 \cos x &\Rightarrow \frac{dy}{dx} = 3 \left[\frac{d}{dx} (\sin x) \right] - 4 \left[\frac{d}{dx} (\cos x) \right] \\ &= 3(\cos x) - 4(-\sin x) \\ &= 3\cos x + 4\sin x \end{aligned}$$

Example 3

Differentiate $y = x^3 \tan x$.Solution: Use the Product Rule: $\frac{dy}{dx} = d(u)(v) = u dv + v du$ Let $u = x^3$ and $v = \tan x$

$$\begin{aligned} \text{We have; } y = x^3 \tan x &\Rightarrow \frac{dy}{dx} = x^3 \left[\frac{d}{dx} (\tan x) \right] + \tan x \left[\frac{d}{dx} (x^3) \right] \\ &= x^3 (\sec^2 x) + \tan x (3x^2) \\ &= x^3 \sec^2 x + 3x^2 \tan x \\ &= x^2 (x \sec^2 x) + 3 \tan x \end{aligned}$$

Factoring



Example 4

Joseph watches an airplane approach at a speed of 500 miles per hour at an altitude of 3 miles. At what rate is the angle of elevation of Joseph's line of sight changing with respect to time when the horizontal distance between the airplane and Joseph is 4 miles?

Solution:

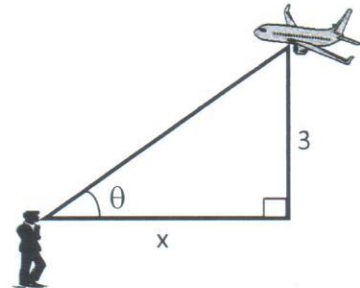
Let x – be the horizontal distance between Joseph and the airplane

t – be the time (in hours)

$\frac{dx}{dt} = -500$ (negative sign indicates that the distance x is decreasing)

We need to find the $\frac{d\theta}{dt}$ when $x = 4$.

From the right triangle in the figure, $\tan \theta = \frac{3}{x}$.



Differentiating both sides with respect to t , we have

$$\tan \theta = \frac{3}{x} \Rightarrow \frac{d}{dx}(\tan \theta) = \frac{d}{dx}\left(\frac{3}{x}\right)$$

Isolating $\frac{d\theta}{dt}$ we have,

$$\begin{aligned} \sec^2 \theta \frac{d\theta}{dt} &= -\frac{3}{x^2} \frac{dx}{dt} \\ \frac{d\theta}{dt} &= -\frac{3}{x^2 \sec^2 \theta} \frac{dx}{dt} \\ \frac{d\theta}{dt} &= -\frac{3}{x^2} \cos^2 \theta \frac{dx}{dt} \end{aligned}$$

We can find the value of $\cos^2 \theta$ from the right triangle in the figure above

when $x = 4$. Hence $\cos^2 \theta = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$

Substituting $x = 4$, $\frac{dx}{dt} = -500$ and $\cos^2 \theta = \frac{16}{25}$ into the formula for $\frac{d\theta}{dt}$

$$\begin{aligned} \text{Thus } \frac{d\theta}{dt} &= -\frac{3}{x^2} \cos^2 \theta \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{3}{(4)^2} \left(\frac{16}{25}\right) (-500) \\ &= -\frac{3}{16} \left(\frac{16}{25}\right) (-500) \\ &= -3(-20) \\ &= 60 \text{ miles per hour} \end{aligned}$$



LEARNING ACTIVITY 12.3.2.7



20 minutes

A. Find the derivatives of the given functions.

1. $y = 10 \sin x + 100$

Answer: _____

2. $f(x) = 8 \sin x \cos x$

Answer: _____

3. $y = 14 \tan x \cos x + 10 \csc x$

Answer: _____

4. $y = \frac{x^5}{4 \sin x}$

Answer: _____

4. $y = 10 \tan(20x)$

Answer: _____



SUMMATIVE TASK 12.3.2



40 minutes

A. In each questions a function is given. Choose the alternative that is the derivative, $\frac{dy}{dx}$, of the function. Circle the letter of your choice.

1. $y = \frac{2-x}{3x+1}$

A. $-\frac{7}{(3x+1)^2}$

B. $-\frac{9}{(3x+1)^2}$

C. $\frac{7}{(3x+1)^2}$

D. $\frac{7-6x}{(3x+1)^2}$

2. $y = (4x+1)^2(1-x)^3$

A. $(4x+1)^2(1-x)^2(5-20x)$

B. $(4x+1)(1-x)^2(4x+11)$

C. $5(4x+1)(1-x)^2(1-4x)$

D. $-24(4x+1)(1-x)^2$

3. $y = \sqrt{3-2x}$

A. $\frac{1}{2\sqrt{3-2x}}$

B. $-\frac{1}{\sqrt{3-2x}}$

C. $-\frac{(3-2x)^{\frac{1}{2}}}{3}$

D. $-\frac{1}{3-2x}$

4. $y = 3x^{\frac{2}{3}} - 4x^{\frac{1}{2}} - 2$

A. $2x^{\frac{1}{3}} - 2x^{-\frac{1}{2}}$

B. $3x^{-\frac{1}{3}} - 2x^{-\frac{1}{2}}$

C. $\frac{2}{x^{\frac{1}{3}}} - \frac{2}{x^{\frac{1}{2}}} - 2$

D. $2x^{-\frac{1}{3}} - 2x^{-\frac{1}{2}}$

5. $y = \sqrt{x^2+2x-1}$

A. $\frac{x+1}{y}$

B. $\frac{1}{2\sqrt{x^2+2x-1}}$

C. $4y(x+1)$

D. none of these



For question 6 – 10, circle the correct answer from the alternatives A, B, C, and D.

6. The equation of the tangent line to the graph of $f(x) = x^3 + 2x + 1$ at the point $(-3, 4)$ is

- A. $y + 4x + 8 = 0$
C. $5y - 4x - 8 = 0$

- B. $2x + y - 4 = 0$
D. $3y + x - 4 = 0$

7. The derivative of the function: $4x^3 + 3x^2 - 8$ is

- A. $14x^2 + 8x$
C. $14x^2 - 8x$

- B. $12x^2 + 6x$
D. $12x^2 - 6x$

8. The slope of the tangent line to the curve $y = 3x^2 - 2x + 4$ at $x = -1$ is represented by

- A. $y' = -8$
C. $y' = -4$

- B. $y' = -6$
D. $y' = -2$

9. Given the function $y = \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}$, then $\frac{d^2y}{dx^2}$ is equal to

A. $\frac{9}{8}x^{-\frac{5}{2}}$

B. $\frac{8}{9}x^{-\frac{5}{2}}$

C. $\frac{9}{8}x^{\frac{2}{5}}$

D. $\frac{8}{9}x^{\frac{2}{5}}$

10. Water leaking onto a floor forms a circular pool. The radius of the pool increases at a rate of 4 cm/min. How fast is the area of the pool increasing when the radius is 5 cm?

- A. $20\pi\text{cm}^2/\text{min}$
C. $40\pi\text{cm}^2/\text{min}$

- B. $30\pi\text{cm}^2/\text{min}$
D. $45\pi\text{cm}^2/\text{min}$

For question 11- 15, differentiate each trigonometric function with respect to x.

11. $f(x) = \sin 2x^3$

Answer: _____

12. $y = \tan 5x^3$

Answer: _____

13. $y = \sec 4x^5$

Answer: _____

14. $f(x) = \sin^3 x^5$

Answer: _____

15. $f(x) = \csc 5x^5$

Answer: _____



2.3.3: CURVE SKETCHING

In the previous section, we have seen the geometrical interpretation of a function as the slope of a curve at a point which is just equal to the slope of the tangent line. We have also discussed the derivative as a rate of change.

In this section, we use the derivative to find the extreme values of a function: the relative maximum and the relative minimum values, which enable us to graph polynomial and rational functions effectively.

12.3.3.1 Maxima and Minima of Algebraic Functions

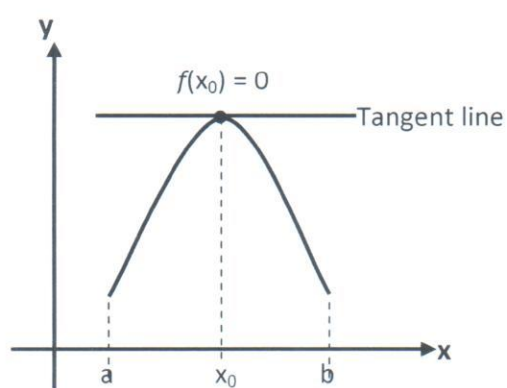
First, we introduce some definitions and theorems to classify the relative maxima and relative minima of functions.

- **The Relative Maximum Value**

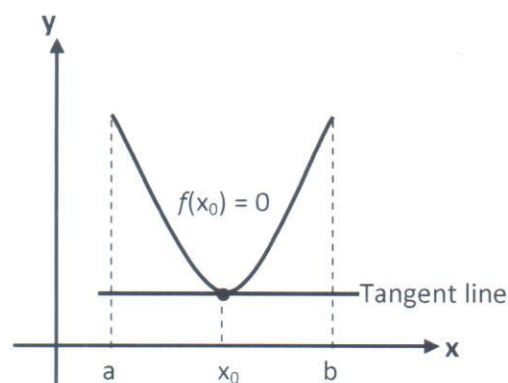
The function $f(x)$ is said to have a **relative maximum value** at x_0 if there exists an open interval containing x_0 , on which $f(x)$ is defined, such that $f(x_0) \geq f(x)$ for all x in this interval (see figure 12.3.3.1a).

- **The Relative Minimum Value**

The function $f(x)$ is said to have a **relative minimum value** at x_0 if there exists an open interval containing x_0 , on which $f(x)$ is defined, such that $f(x_0) \leq f(x)$ for all x in this interval (see figure 12.3.3.1b).



a. Relative maximum value at x_0



b. Relative Minimum value at x_0

Figure 12.3.3.1: Relative Extrema

- **The Relative Extremum**

A function $f(x)$ is said to have a **relative extremum** at x_0 if it has either a relative maximum or relative minimum at x_0 . (see Figure 12.3.3.1).



- **The Critical Number**

The **critical number** of a function $f(x)$ is any value of x in the domain of $f(x)$ at which $f'(x)$ is equal to 0 or at which $f(x)$ is not differentiable. The **relative extrema** of a function, if any, occur at the **critical numbers**.

- **The Extreme Value Theorem**

If a function f is continuous on a closed interval $[a, b]$, then f has an **absolute maximum** and an **absolute minimum value** at least once in $[a, b]$.

- **The Absolute Maximum Value**

$f(x_0)$ is said to be the **absolute maximum value** of a function $f(x)$ if x_0 is in the domain of $f(x)$ and $f(x_0) \geq f(x)$ for all values of x in the domain of $f(x)$. (see figure 12.3.3.2).

- **The Absolute Minimum Value**

$f(x_0)$ is said to be the **absolute minimum value** of a function $f(x)$ if x_0 is in the domain of $f(x)$ and $f(x_0) \leq f(x)$ for all values of x in the domain of $f(x)$. (See figure 12.3.3.2).

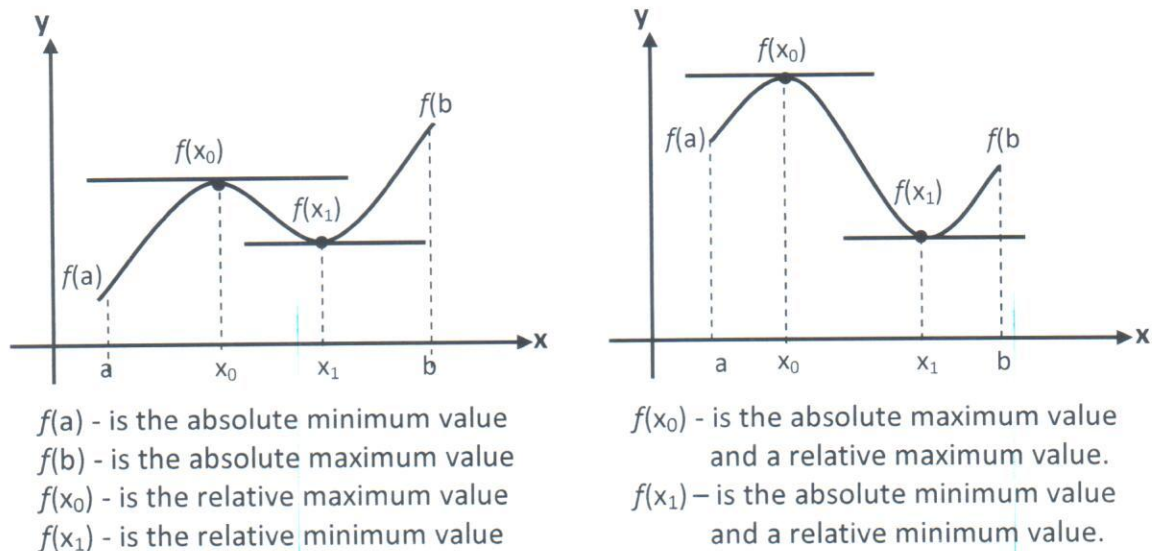


Figure 12.3.3.2: Absolute Extrema of Functions

Strategy for obtaining the absolute extrema of a function

1. Find all the critical number of $f(x)$ on (a, b) .
2. Calculate $f(x_0)$ for each of the critical number x_0 .
3. Find the values of $f(a)$ and $f(b)$.
4. The largest of the values from steps 1 to 3 is the absolute maximum value, and the smallest of the values is the absolute minimum value.



Example 1

If $y = f(x) = \frac{x^3}{4} - 3x$, find the absolute maximum and minimum values of f on the closed interval $[-3, 3]$. Sketch the graph.

Solution:

Step 1: Find the critical numbers of f . We do this by differentiating the given function with respect to x .

$$\text{We get } y = f(x) = \frac{x^3}{4} - 3x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{4} - 3$$

$$\text{Solving } \frac{dy}{dx} = 0 \text{ for } x, \text{ we have } \frac{3x^2}{4} - 3 = 0 \quad \Rightarrow \quad x = 2, -2$$

Thus, the **critical numbers** are $x_1 = 2$ and $x_2 = -2$.

Step 2: Find the values of $f(a)$ and $f(b)$.

Since f is continuous on $[-3, 3]$, it follows from our discussion that the absolute maximum and minimum are among the values $f(-3)$, $f(-2)$, $f(2)$ and $f(3)$. Calculating these values, we have

$$f(-3) = \frac{(-3)^3}{4} - 3(-3) = \frac{-27}{4} + 9 = -6.75 + 9 = 2.25 \quad \Rightarrow (-3, 2.25)$$

$$f(-2) = \frac{(-2)^3}{4} - 3(-2) = \frac{-8}{4} + 6 = -2 + 6 = 4 \quad \Rightarrow (-2, 4)$$

$$f(2) = \frac{(2)^3}{4} - 3(2) = \frac{8}{4} + 6 = 2 - 6 = -4 \quad \Rightarrow (2, -4)$$

$$f(3) = \frac{(3)^3}{4} - 3(3) = \frac{27}{4} - 9 = 6.75 - 9 = -2.25 \quad \Rightarrow (3, -2.25)$$

Therefore, the absolute minimum point of $y = f(x) = \frac{x^3}{4} - 3x$ on $[-3, 3]$ is the point $(2, -4)$ and it is a **relative minimum point** and the point $(-2, 4)$ is absolute maximum point and it is also the **relative maximum point**.

To sketch the graph, we make the table of values:

x	-3	-2	-1	0	1	2	3
y	2.25	4	2.75	0	-2.75	-4	-2.25

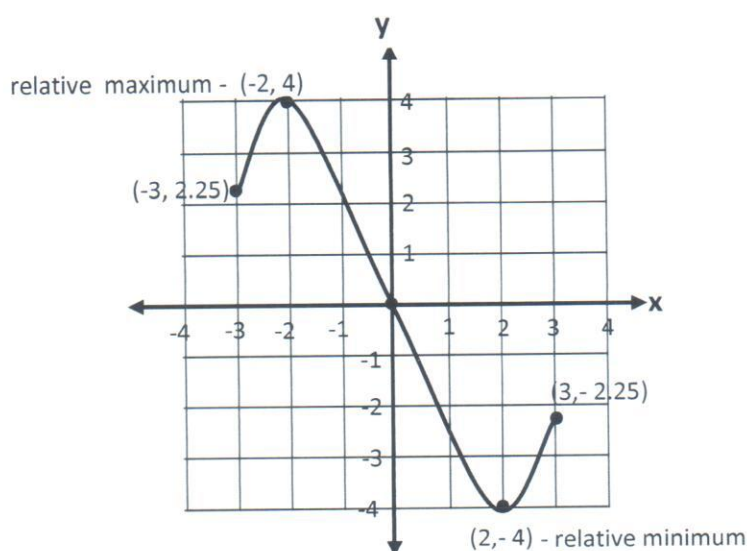


Figure 12.3.3.3: Absolute Extrema of $f(x) = \frac{x^3}{4} - 3x$.

Example 2

Find the absolute maximum and minimum values of $y = x^3 - 3x + 2$ on the closed interval $[-2, 2]$. Sketch the graph.

Solution:

Differentiate $y = x^3 - 3x + 2$ with respect to x . $\frac{dy}{dx} = 3x^2 - 3$

Solve for $\frac{dy}{dx}$, we have $3x^2 - 3 = 0$
 $x = \pm 1$

Thus the critical numbers are $x_1 = 1$ and $x_2 = -1$. Since f is continuous on $[-2, 2]$, it follows from our discussion that the absolute maximum and minimum are among the values $f(-2)$, $f(-1)$, $f(1)$ and $f(2)$. Calculating these values, we have

$$\begin{aligned} f(-2) &= (-2)^3 - 3(-2) + 2 = -8 + 6 + 2 = 0 && \Rightarrow (-2, 0) \\ f(-1) &= (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4 && \Rightarrow (-1, 4) \\ f(1) &= (1)^3 - 3(1) + 2 = 1 - 3 + 2 = 0 && \Rightarrow (1, 0) \\ f(2) &= (2)^3 - 3(2) + 2 = 8 - 6 + 2 = 4 && \Rightarrow (2, 4) \end{aligned}$$

The relative maximum occurs at $(-1, 4)$ and the relative minimum occurs at $(1, 0)$.

Notice that the relative extrema are equal to the endpoints extrema. (See the graph on Figure 12.3.3.4 at the next page).

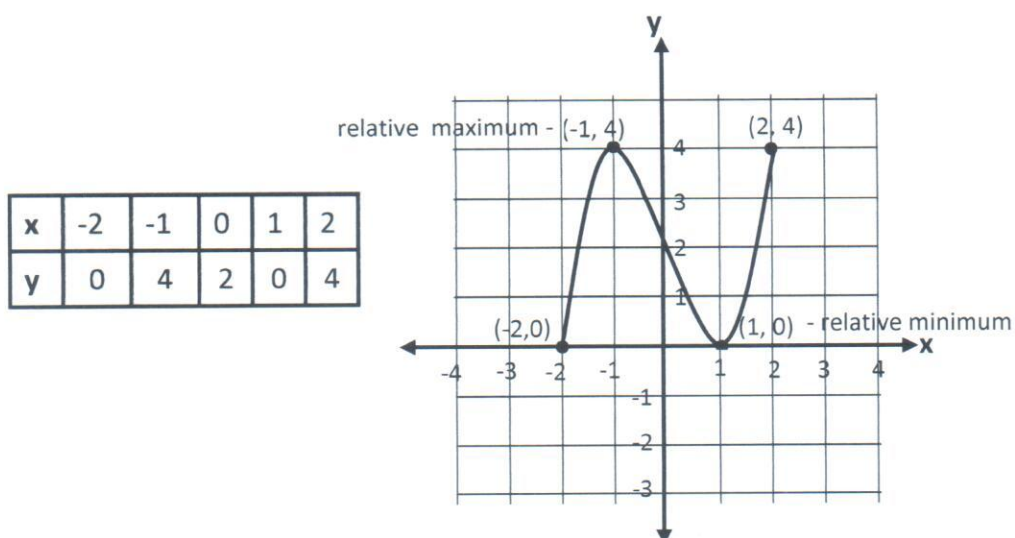


Figure 12.3.3.4: Relative Extrema are equal to Endpoints Extrema

Example 3

Find the absolute maximum and minimum values of $y = f(x) = 5$ if they exist over the interval $[-100, 100]$. Sketch the graph.

Solution:

Differentiating $f(x) = 5$, with respect to x , $\frac{dy}{dx} = 0$. This implies a constant function.

In this case, $\frac{dy}{dx} = 0$ for any value of x .

Therefore there is neither a minimum or maximum value.

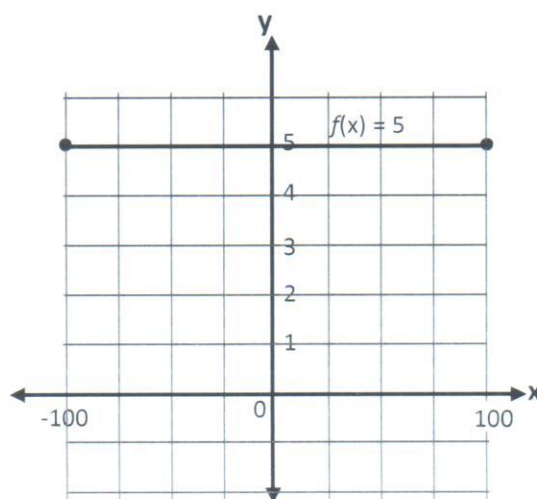


Figure 12.3.3.5:



Example 4

Find the absolute maximum and minimum values of $y = f(x) = (x + 2)^3 - 1$, if they exist over the interval $[-3, 0]$. Sketch the graph.

Solution:

$$\text{Find } \frac{dy}{dx}: \quad f(x) = (x + 2)^3 - 1 \quad \Rightarrow \frac{dy}{dx} = 3(x + 2)^2$$

$$\text{Find the critical points by setting } \frac{dy}{dx} = 0. \quad \frac{dy}{dx} = 3(x + 2)^2 = 0$$

Solve for x .

$$\begin{aligned} 3(x + 2)^2 &= 0 \\ (x + 2)^2 &= 0 \\ x + 2 &= 0 \\ x &= -2 \end{aligned}$$

Thus the critical numbers are $x_1 = -2$. Since f is continuous on $[-3, 0]$, it follows from our discussion that the absolute maximum and minimum are among the values $f(-3)$, $f(-2)$, $f(-1)$ and $f(0)$. Calculating these values, we have

$$f(-3) = (-3 + 2)^3 - 1 = (-1)^3 - 1 = -2 \quad \Rightarrow (-3, -2)$$

$$f(-2) = (-2 + 2)^3 - 1 = (0)^3 - 1 = -1 \quad \Rightarrow (-2, -1)$$

$$f(-1) = (-1 + 2)^3 - 1 = (1)^3 - 1 = 0 \quad \Rightarrow (-1, 0)$$

$$f(0) = (0 + 2)^3 - 1 = (2)^3 - 1 = 7 \quad \Rightarrow (0, 7)$$

x	-3	-2	-1	0	1
y	-2	-1	0	7	26

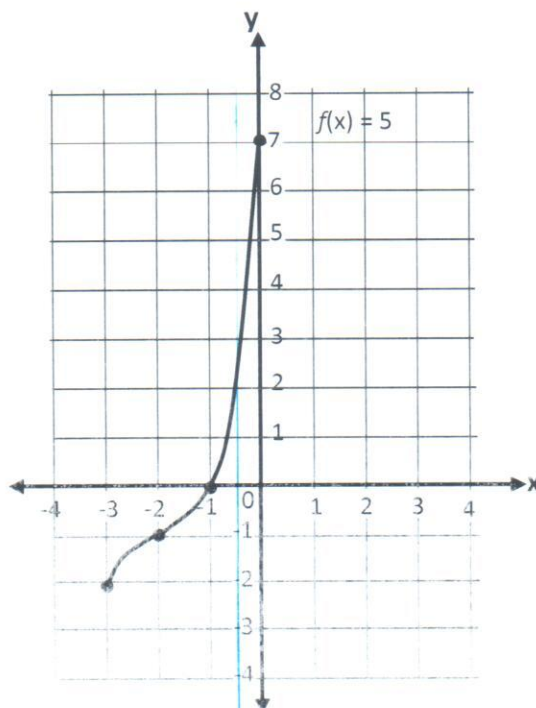


Figure 12.3.3.5:



LEARNING ACTIVITY 12.3.3.1



20 minutes

A. Find the critical points of $y = f(x)$.

1. $y = x^2 - 6x + 9$

Answer: _____

2. $y = x^3 - 3x + 4$

Answer: _____

3. $y = (x + 2)^2$

Answer: _____

B. Find the absolute minimum and absolute maximum on the indicated interval.

1. $y = x^3 - 6x^2 - 9x + 3$; $[-3, -1]$

Answer: _____

2. $y = x^3 - 3x^2 - 3$; $[0, 3)$

Answer: _____



12.3.3.2: Applied Optimization Problems

One of the most important applications of derivatives is in the solving of optimization problems. To **optimize** something means to make it as useful or effective as possible.

We will use the Extreme Value Theorem to solve applied optimization problems. Such type of problems can be solved using the techniques for finding the absolute maximum or minimum of a function.

In this section we are going to look at optimization problems. In optimization problems we are looking for the largest value or the smallest value that a function can take.

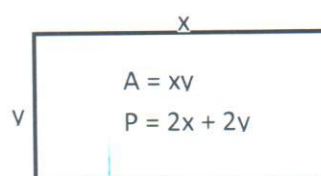
Strategy for Solving Optimization Problems

1. Read the problem carefully. Assign variables to all quantities involved in the problem. When necessary, make a sketch and label it appropriately.
2. Translate all relationships into mathematical equations, then identify the variables to be maximized or minimized and express it as function of only one variable.
3. Determine the values of the variables for which the stated problem make sense.
4. Apply the Extreme Value Theorem to determine the desired maximum or minimum.

Example 1 (Finding the maximum area)

Find the dimensions of the rectangular lot that can be enclosed using 1000 m of fencing material and whose area is as large as possible

Solution: We draw the figure and label the desired quantities.



We let x = length of the rectangular lot
 y = width of the rectangular lot
 A = area of the rectangular lot
 P = perimeter of the rectangular lot

We need to find the dimensions of the rectangular lot whose area is as large as possible with a given perimeter.

The area of the lot is given by the formula $A = xy$

Since the perimeter of the lot is 1000 m, we relate the variables x and y by the equation

$$P = 2x + 2y \quad \Rightarrow \quad 2x + 2y = 1000$$
$$y = 500 - x$$

Substitute $y = 500 - x$ into the formula $A = xy$

We have $A = x(500 - x) \Rightarrow A = 500x - x^2$



Differentiating $A = 500x - x^2$ with respect to x then solving for x , we get

$$\frac{dy}{dx} = 500 - 2x \Rightarrow 500 - 2x = 0 \Rightarrow x = 250 \text{ m}$$

Substituting $x = 250$ into $y = 500 - x$, we get

$$y = 500 - 250$$

$$y = 250 \text{ m}$$

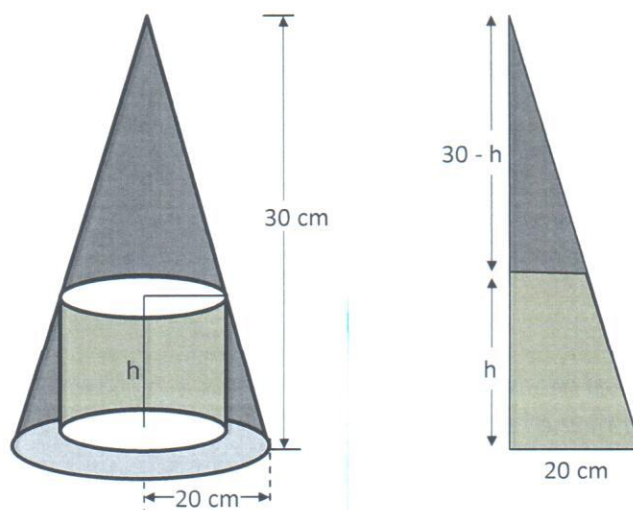
Since the two perpendicular sides are equal, we say that the rectangular lot formed a square with sides equal to 250 m. We can solve the maximum area using the formula $A = xy$. Hence,

$$A = 250(250) = \mathbf{62\,500 \text{ square metres.}}$$

Example 2 (Finding the largest volume)

Find the radius and height of the right-circular cylinder of largest volume that can be inscribed in a right-circular cone with radius 20 cm and height 30 cm.

Solution: We draw the figure and label the desired quantities.



We let r = radius of the right-circular cylinder

h = height of the right-circular cylinder

V = volume of the right-circular cylinder

The volume V of a right-circular cylinder is given by $V = \pi r^2 h$

To express the relationship between r and h , we use similar triangles and solving for h , we have

$$\frac{30-h}{r} = \frac{30}{20}$$

$$2(30 - h) = 3r$$



$$h = \frac{60-3r}{2}$$

Substituting the value of h into the formula $V = \pi r^2 h$, we get

$$V = \pi r^2 \left(\frac{60-3r}{2} \right) \Rightarrow V = 30\pi r^2 - \frac{3}{2}\pi r^3; \quad 0 < r \leq 20$$

Differentiating $V = 30\pi r^2 - \frac{3}{2}\pi r^3$ with respect to r , we get

$$\frac{dV}{dr} = 60\pi r - \frac{9}{2}\pi r^2$$

$$\text{Solving } \frac{dV}{dr} = 0 \text{ for } r, \text{ we obtain } 60\pi r - \frac{9}{2}\pi r^2 = 0 \Rightarrow \frac{1}{2}\pi r(120-9r) = 0$$

Setting each factor to 0, then solving for r , we get

$$\frac{1}{2}\pi r = 0 \Rightarrow r_1 = 0$$

$$120 - 9r = 0 \Rightarrow r_2 = 13.33 \text{ cm}$$

Hence, the critical numbers are $r_1 = 0$ and $r_2 = 13.33$ cm, and $r_3 = 20$ cm. Substituting these values into the formula $V = 30\pi r^2 - \frac{3}{2}\pi r^3$, we get

Since $r > 0$, $r_1 = 0$ is rejected.

$$\begin{aligned} \text{So when } r_2 = 13.33 \text{ cm, } V &= 30\pi(13.33)^2 - \frac{3}{2}\pi(13.33)^3 \\ &= 5330.667\pi - 3552.889\pi \\ &= 1777.78 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \text{when } r_3 = 20 \text{ cm, } V &= 30\pi(20)^2 - \frac{3}{2}\pi(20)^3 \\ &= 12000\pi - 12000\pi \\ &= 0 \end{aligned}$$

To solve for h , we use the equation $h = \frac{60-3r}{2}$

$$h = \frac{60-3(13.33)}{2}$$

$$h = 10.005 \text{ or } 10.01 \text{ cm}$$

Thus the **maximum volume** $V = 1777.78 \text{ cm}^3$ is obtained when the inscribed cylinder has radius $r = 13.33$ cm and $h = 10.01$ cm.

**LEARNING ACTIVITY 12.3.3.2**

20 minutes

Solve the following problems.

1. Find two non-negative numbers so that their sum is 100 and the sum of their squares is maximum.

Answer: _____

2. Given a rectangle with perimeter 864. Find the largest possible area.

Answer: _____

3. A farmer has 800 m of fencing material to enclose a rectangular pen adjacent to a long existing wall. He will use the wall for one side of the pen and the available fencing material for the remaining three sides. What is the maximum area that can be enclosed this way?

Answer: _____

4. Find the lengths of the sides of a triangle whose two sides lie on the coordinate axes and the other side passes through the point (1, 1).

Answer: _____



12.3.3.3: The First Derivative Test

Without calculus we do not know the exact behaviour of the curve between the points we plot. Even if we plot hundreds of points and connect them with line segments, the maximum and minimum values of the curve could not be determined accurately.

In this section, we will use the derivative to determine the behaviour of a continuous function on a certain interval whether it is increasing or decreasing and to find the relative extrema of a function on that interval.

Increasing and Decreasing Functions

Definition

If a graph of a function **rises from left to right** over an interval (I), it is said to be **increasing** over that interval. If the graph **drops from left to right** over an interval (I), it is said to be **decreasing** over that interval.

A function $f(x)$ defined on an interval is said to be **increasing** on that interval if and only if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$

where x_1 and x_2 are any numbers in the interval .

A function $f(x)$ defined on an interval is said to be **decreasing** on that interval if and only if

$$f(x_1) > f(x_2) \text{ whenever } x_1 > x_2$$

where x_1 and x_2 are any numbers in the interval .

Examples graphs of increasing function

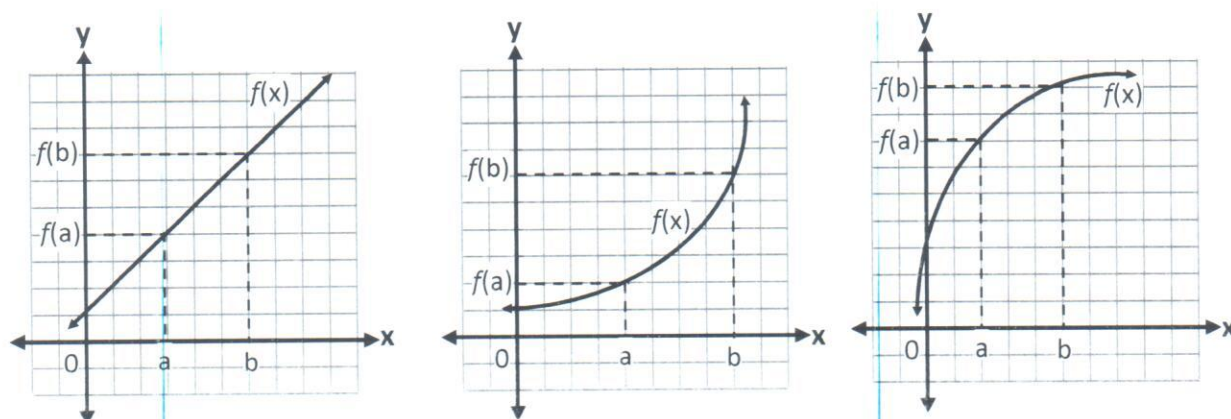
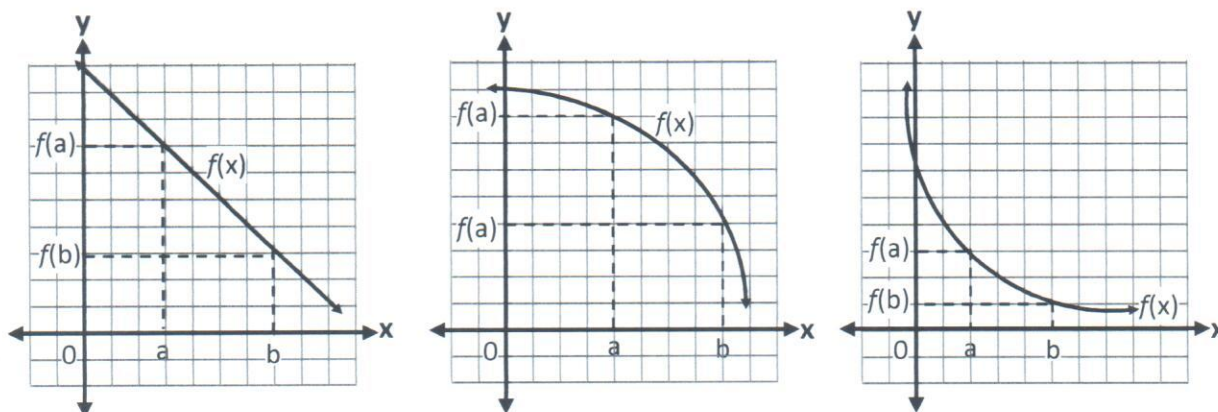


Figure 12.3.3.6: f is increasing on $[a, b]$

**Examples graphs of decreasing function****Figure 12.3.3.7: f is decreasing on $[a, b]$**

Derivatives can be used to determine whether a function is increasing or decreasing.

Theorem

Let $f(x)$ be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) ,

- If $f'(x) > 0$ for every x in (a, b) , then $f(x)$ is **increasing** on $[a, b]$.
- If $f'(x) < 0$ for every x in (a, b) , then $f(x)$ is **decreasing** on $[a, b]$.
- If $f'(x) = 0$ for every x in (a, b) , then $f(x)$ is **constant** on $[a, b]$.

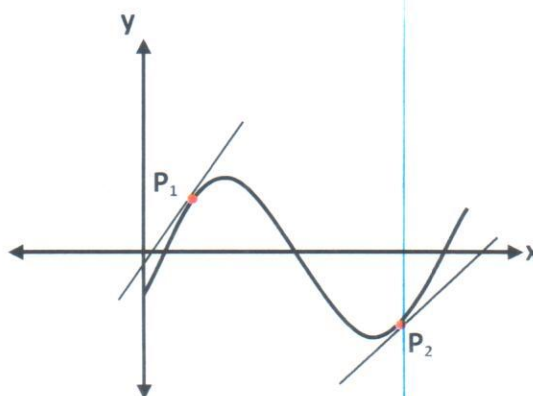
The sign of the Differential Coefficient

The sign of $\frac{dy}{dx}$ at different points on the graph of $y = f(x)$ can give a great deal of information about the function which can be used to investigate the shape of its graph.

- If $\frac{dy}{dx}$ is positive at a point P on a curve, then the tangent is positive at that point and the function is said to be **increasing function** at P . That is the function is increasing as x increases.

$$\frac{dy}{dx} > 0 \text{ at } P_1 \text{ and } P_2$$

The function increases as x increases at these points.

**Figure 12.3.3.8**



2. If $\frac{dy}{dx}$ is negative at a point P on a curve, then the tangent is negative at that point and the function is said to be **decreasing function** at P. That is the function is decreasing as x increases.

$$\frac{dy}{dx} > 0 \text{ at } P_1 \text{ and } P_2$$

The function decreases as x increases at these points.

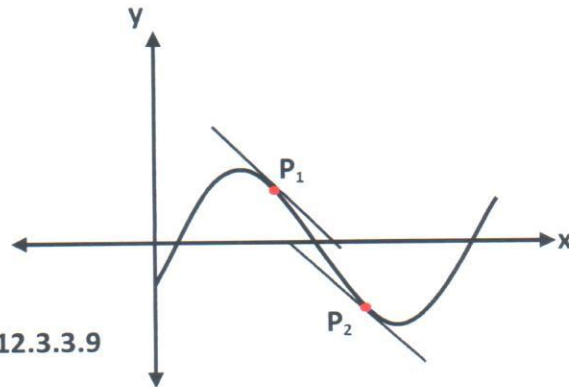


Figure 12.3.3.9

2. If $\frac{dy}{dx} = 0$ at a point P on a curve, then the tangent is parallel to the x-axis at that point and the graph is said to have a **stationary point** at P.

$$\frac{dy}{dx} = 0 \text{ at } P_5 \text{ and } P_6$$

These are stationary points.

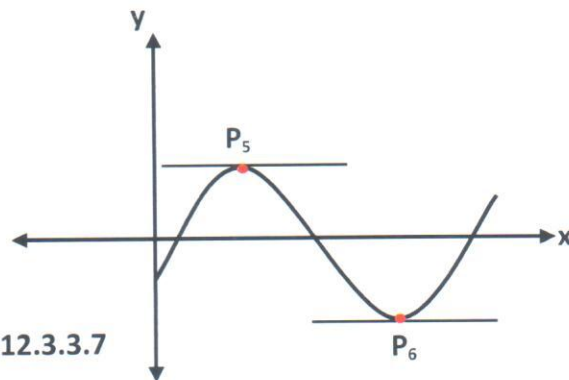


Figure 12.3.3.7

Example 1

Given $y = f(x) = x^3 + 6x^2 + 9x$

- a. Find the open intervals where the given function is increasing or decreasing and graph the function.

Solution: Find $\frac{dy}{dx}$: $y = f(x) = x^3 + 6x^2 + 9x \Rightarrow \frac{dy}{dx} = f'(x) = 3x^2 + 12x + 9$

Setting $\frac{dy}{dx}$ or $f'(x) = 0$, we have $3x^2 + 12x + 9 = 0$

Factor and solve for x.

$$3(x^2 + 4x + 3) = 0$$

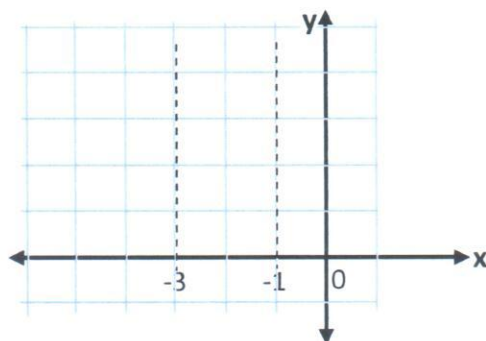
$$3(x + 3)(x + 1) = 0$$

$$x = -3; x = -1$$

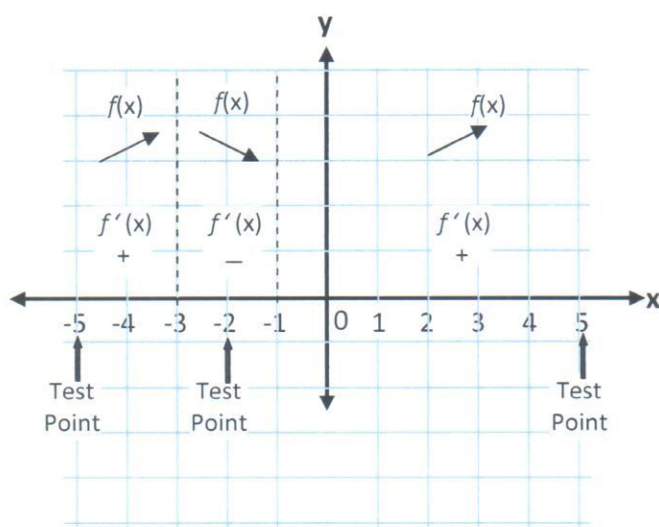
Since there is no value of x where $f'(x)$ fails to exist, the only critical numbers are -3 and -1 . Locate $x = -3$ and $x = -1$ on the number line. The points determine the three intervals $(-\infty, -3)$, $(-3, -1)$ and $(-1, \infty)$.



Graph the intervals.



The behaviour of the graph in each interval



Now choose any value x in the interval $(-\infty, 3)$.

When $x = -5$, $\frac{dy}{dx} = f'(x) = 3x^2 + 12x + 9$

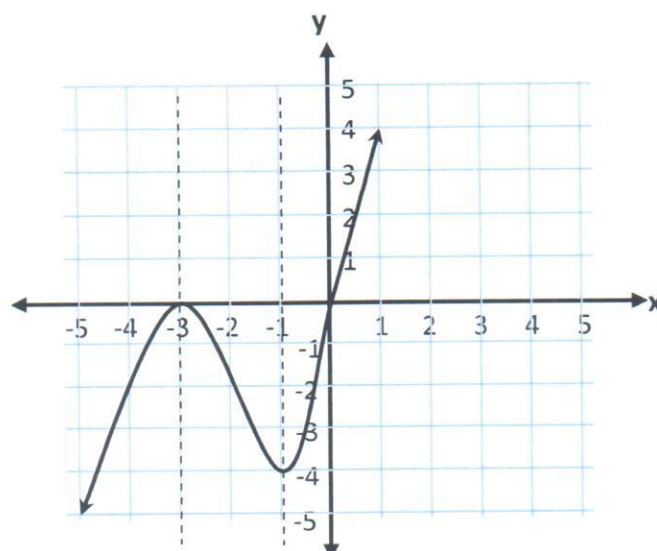
$f'(-5) = 3(-5)^2 + 12(-5) + 9 = 3(25) - 60 + 9 = 75 - 60 + 9 = 24$, which is positive. Since one value of x in this interval makes $f'(x) > 0$, all values will make it so, and therefore f is increasing on $(-\infty, 3)$.

When $x = -2$, then $f'(-2) = 3(-2)^2 + 12(-2) + 9 = 3(4) - 24 + 9 = 12 - 24 + 9 = -3$, which is negative. This means that f is decreasing on $(-3, -1)$.

When $x = 5$, then $f'(5) = 3(5)^2 + 12(5) + 9 = 3(25) + 60 + 9 = 75 + 60 + 9 = 144$, which is positive. This means that f is increasing on $(-1, \infty)$.

We learned that horizontal line has a slope of 0. Horizontal tangents can be found by solving for $f'(x) = 0$. As shown above, $f'(x) = 0$ when $x = -3$ and -1 .

The graph of the function $y = f(x) = x^3 + 6x^2 + 9x$ is shown on the next page.

Graph of $y = f(x) = x^3 + 6x^2 + 9x$ **Example 2**

Given the function $y = f(x) = 2x^3 - 15x^2 + 36x - 24$. Determine the values of x where the function is increasing and decreasing.

Solution:

Find $\frac{dy}{dx}$ or $f'(x)$: $y = f(x) = 2x^3 - 15x^2 + 36x - 24 \Rightarrow \frac{dy}{dx} = f'(x) = 6x^2 - 30x + 36$

Find the critical numbers by setting $f'(x) = 0$ and solve the resulting equation by factoring.

We have $\frac{dy}{dx} = f'(x) = 6x^2 - 30x + 36 \Rightarrow 6x^2 - 30x + 36 = 0$

$$6(x^2 - 5x + 6) = 0$$

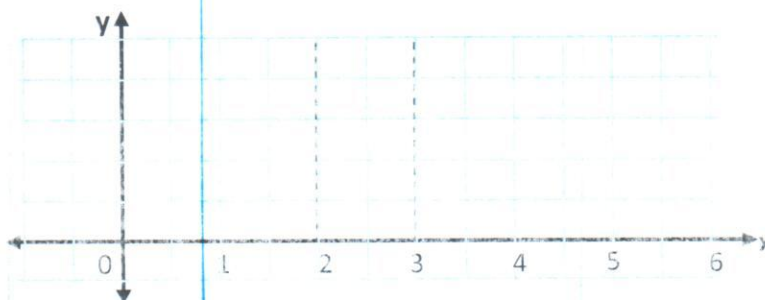
$$6(x - 2)(x - 3) = 0$$

$$x - 2 = 0; x - 3 = 0$$

$$x = 2; x = 3$$

The tangents are horizontal at $x = 2$ and $x = 3$. There is no value of x where $f'(x)$ fails to exist. Hence, the only critical numbers are 2 and 3. To determine where the function is increasing or decreasing, test a point on each interval as partitioned by $x = 2$ and $x = 3$.

Below is the graph of the intervals.





When $x = 0$, a point in the interval $(-\infty, 2)$: $f'(x) = 6x^2 - 30x + 36$

$$f'(0) = 6(0)^2 - 30(0) + 36 = 0 - 0 + 36 = 36, \text{ which is positive.}$$

Therefore, the graph is increasing on $(-\infty, 2)$.

When $x = 2.5$, a point in the interval $(2, 3)$:

$$f'(2.5) = 6(2.5)^2 - 30(2.5) + 36 = 37.5 - 75 + 36 = -1.5, \text{ which is negative.}$$

Therefore, the graph is decreasing on $(2, 3)$.

When $x = 4$, a point in the interval $(3, \infty)$:

$$f'(4) = 6(4)^2 - 30(4) + 36 = 96 - 120 + 36 = 12, \text{ which is positive.}$$

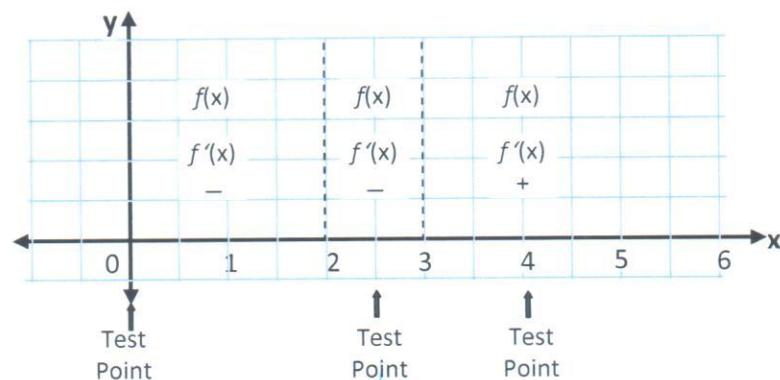
Therefore, the graph is increasing on $(3, \infty)$.

The slope of the tangent line is 0 at $x = 2$ and at $x = 3$.

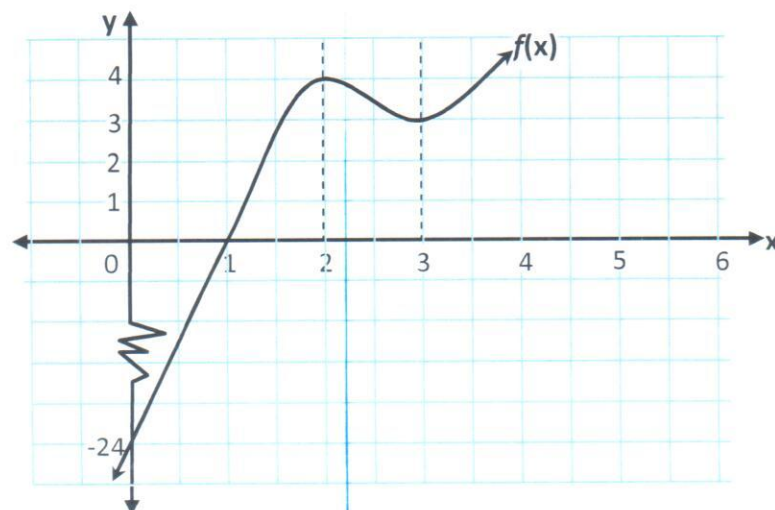
Note that, when $x = 2$, $f(x) = 2x^3 - 15x^2 + 36x - 24 \Rightarrow f(2) = 2(2)^3 - 15(2)^2 + 36(2) - 24 = 4$

and when $x = 3$, $\Rightarrow f(3) = 2(3)^3 - 15(3)^2 + 36(3) - 24 = 3$.

The behaviour of the graph in each interval is as follows:



The graph of $f(x) = 2x^3 - 15x^2 + 36x - 24$ is shown below.



**LEARNING ACTIVITY 12.3.3.3**

20 minutes

1. Given the function $y = f(x) = x^2 - 5x + 6$
 - a. Find the critical numbers.
 - b. Find the relative extrema.
 - c. Describe the intervals in which it is increasing or decreasing.
 - d. Sketch the graph.

2. Consider the function $y = f(x) = x^3 - 3x^2 + 6$.
 - a. Find all the values of x for which f is increasing.
 - b. Find all the values of x for which f is decreasing.
 - c. Find all values of x for which f has a relative maximum.
 - d. Find all values of x for which f has a relative minimum.
 - e. Sketch the graph.



12.3.3.4 Concavity and the Second Derivative Test

We have observed the behaviour of the function $f(x)$ as it increases or decreases on a certain interval using the first derivative test.

In this section, we will extend this idea by locating the interval in which the graph of $f(x)$ is concave upward or concave downward.

The Concavity

Let $f(x)$ is a function that is differentiable on an open interval L . The graph of $f(x)$ is defined as

- Concave upward** on L if $f'(x)$ is increasing on L , or the graph of $f(x)$ is above the tangent line on every point on that interval (see Figure 12.3.3.4.1a)
- Concave downward** on L if $f'(x)$ is decreasing on L , or the graph of $f(x)$ is below the tangent line on every point on that interval (see Figure 12.3.3.4.1b)

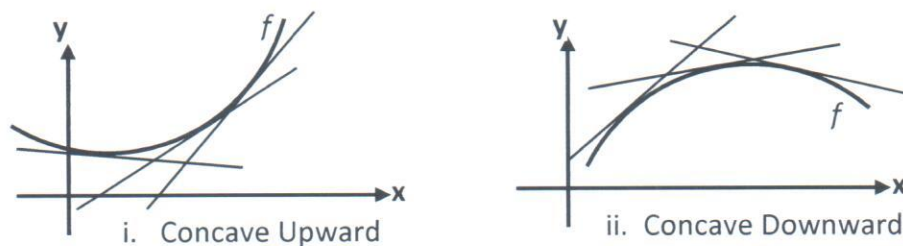


Figure 12.3.3.4.1: Graphical Representation of Concavity

The Test of Concavity

Let $f(x)$ is a function whose second derivative exists on an open interval L .

- If $f''(x) > 0$ for all x in L , then the graph of $f(x)$ is **concave upward** on L .
- If $f''(x) < 0$ for all x in L , then the graph of $f(x)$ is **concave downward** on L .

The Point of Inflection

If the graph of a continuous function $f(x)$ possesses a tangent line at a point where its concavity changes from upward to downward (or vice versa), then the point is called a **point of inflection**, (See Figure 12.3.3.4.2)

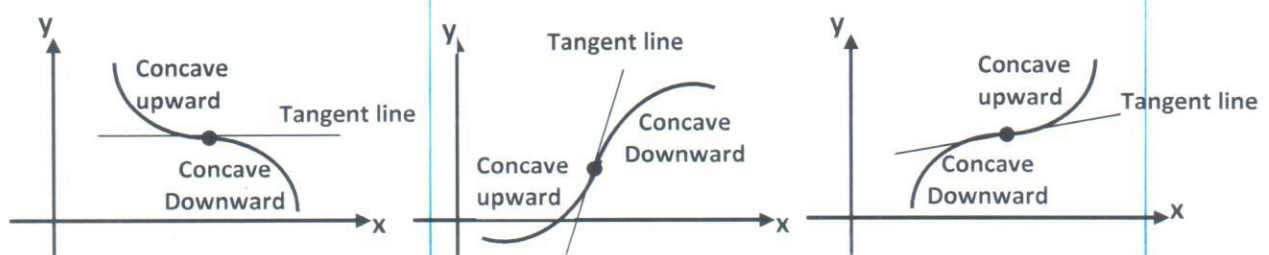


Figure 12.3.3.4.2: Types of Points of Inflection

**Note:**

At a **point of inflection** the graph crosses its tangent line. It occurs where the concavity of the graph changes. Thus, to locate the possible points of inflection, we need only to determine the value of x for which $f'(x) = 0$ or for which $f'(x)$ does not exist.

The Second Derivative Test

Let $f'(x)$ exist on some open interval containing x_0 and $f'(x_0) = 0$.

- If $f''(x_0) > 0$, then $f(x_0)$ is a **relative minimum value**.
- If $f''(x_0) < 0$, then $f(x_0)$ is a **relative maximum value**.
- If $f''(x_0) = 0$, then the test fails.

Example 1

Use the Second Derivative Test to find the relative extrema, discuss concavity, points of inflection, and sketch the graph of $y = x^3 - 3x + 3$, x is within the interval $[-2, 2]$.

Solution: We start by finding the critical numbers of the given function.

Differentiating $y = x^3 - 3x + 3$ with respect to x , we get

$$y = x^3 - 3x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 - 3$$

Setting $\frac{dy}{dx} = 0$ and solving for x , we obtain the critical numbers: $x = -1$ and $x = 1$.

Using the Second Derivative Test, we get $\frac{d^2y}{dx^2} = 6x$

$$\text{When } x = -1, f(-1) = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5 \quad \Rightarrow (-1, 5)$$

$$\frac{d^2y}{dx^2} = 6x = 6(-1) = -6 < 0$$

Thus, the point $(-1, 5)$ is a relative maximum point.

$$\text{When } x = 1, f(1) = (1)^3 - 3(1) + 3 = 1 - 3 + 3 = 1 \quad \Rightarrow (1, 1)$$

$$\frac{d^2y}{dx^2} = 6x = 6(1) = 6 > 0$$

Thus, the point $(1, 1)$ is a relative minimum point.

To find the possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$ for x .

$$\text{Hence} \quad 6x = 0 \quad \Rightarrow \quad x = 0.$$

$$\text{When } x = 0, f(0) = (0)^3 - 3(0) + 3 = 3 \quad \Rightarrow (0, 3)$$

Thus, $(0, 3)$ is a point of inflection.



We summarize the results in Table 12.3.3.4.1.

Interval	$(-2, 0)$		$(0, 2)$
Critical Numbers	-1		1
Point of Inflection		0, 3	
Value of $f(x)$	5	3	1
Value of $f'(x)$	0	-3	0
Sign of $f'(x)$	-	0	+
Graph of $f(x)$	Concave upward		Concave Downward
Conclusion	Relative maximum		Relative Minimum

Now let us sketch the graph:

Use the table of values.

x	-2	-1	0	1	2
y	-1	5	3	1	5

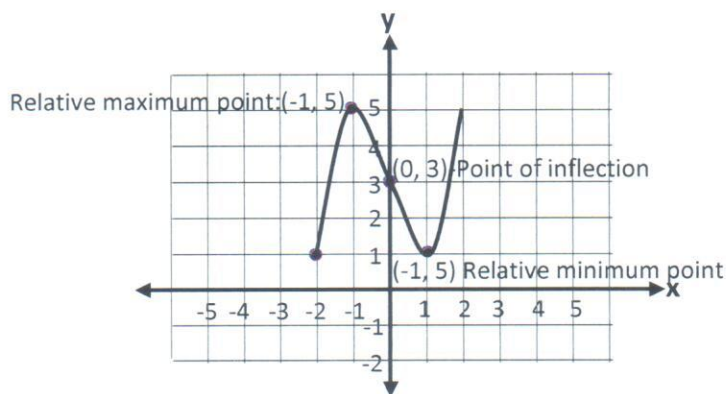


Figure 12.3.3.4.3: graph of $y = x^3 - 3x + 3$

Example 2

Use the Second Derivative Test to find the relative extrema, discuss concavity, points of inflection, and sketch the graph of $y = \frac{1}{4}x^4 - 2x^2 + 4$, x is within the interval $[-2, 2]$.

Solution: We start by finding the critical numbers of the given function.

Differentiating $\frac{1}{4}x^4 - 2x^2 + 4$ with respect to x , we get

$$\frac{1}{4}x^4 - 2x^2 + 4 \Rightarrow \frac{dy}{dx} = 4\left(\frac{1}{4}\right)x^3 - 2(2)x$$

$$\frac{dy}{dx} = x^3 - 4x$$

Setting $\frac{dy}{dx} = 0$ and solving for x . $x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0$

$$x(x + 2)(x - 2) = 0$$

We obtain the critical numbers: $x = 0$, $x = -2$ and $x = 2$.



Using the Second Derivative Test, we get $\frac{d^2y}{dx^2} = 3x^2 - 4$

Evaluating the critical numbers using the Second Derivative Test:

When $x = 0$, $\frac{d^2y}{dx^2} = 3x^2 - 4 = 3(0)^2 - 4 = -4 < 0$

Hence, the point $(0, 4)$ is a relative maximum point.

When $x = -2$, $\frac{d^2y}{dx^2} = 3x^2 - 4 = 3(-2)^2 - 4 = 12 - 4 = 8 > 0$

Hence, the point $(-2, 0)$ is a relative minimum point.

When $x = 2$, $\frac{d^2y}{dx^2} = 3x^2 - 4 = 3(2)^2 - 4 = 12 - 4 = 8 > 0$

Hence, the point $(2, 0)$ is a relative minimum point.

To find the possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$ for x .

Hence $3x^2 - 4 = 0 \Rightarrow x = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{\sqrt{3}} = \pm 1.15$

Thus, we have two points of inflection: $(-1.15, 1.79)$ and $(1.15, 1.79)$.

We summarize the results in Table 12.3.3.4.2.

Interval	$x \leq -1.15$	$(-1.15, 1.15)$	$x \geq 1.15$
Critical Numbers	-2	0	2
Point of Inflection		-1.15	1.15
Value of $f(x)$	0	1.79	4
Value of $f'(x)$	0	+	-
Sign of $f'(x)$	+	0	+
Graph of $f(x)$	Concave upward	Concave Downward	Concave upward
Conclusion	Relative minimum	Relative maximum	Relative minimum

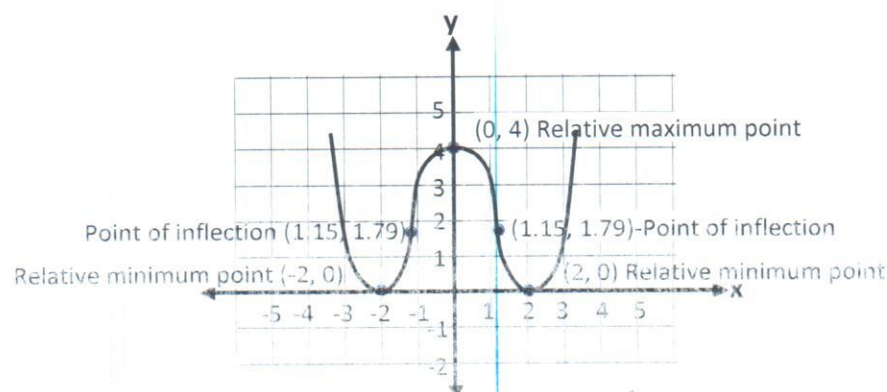


Figure 12.3.3.4.4: The graph of $y = \frac{1}{4}x^4 - 2x^2 + 4$



Example 3

Given the function $y = f(x) = 3x^5 - 5x^3$, use the Second Derivative Test to find the relative extrema, concavity, and points of inflection. Sketch the graph of f .

Solution:

Differentiating y with respect to x , then solving the critical numbers, we obtain

$$y = f(x) = 3x^5 - 5x^3 \Rightarrow \frac{dy}{dx} = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$$

$$\begin{aligned} \text{Setting } \frac{dy}{dx} = 0 &\Rightarrow 15x^2(x^2 - 1) = 0 \\ &15x^2 = 0; x + 1 = 0; x - 1 = 0 \\ &x = 0; -1 \text{ and } 1 \end{aligned}$$

Differentiating the first derivative again, we have

$$\frac{dy}{dx} = 15x^4 - 15x^2 \Rightarrow \frac{d^2y}{dx^2} = 60x^3 - 30x$$

Evaluating the critical numbers using the Second Derivative Test, we get

$$\text{when } x = 0 \Rightarrow \frac{d^2y}{dx^2} = 60x^3 - 30x = 60(0)^3 - 30(0) = 0, \text{ the test fails.}$$

$$\text{when } x = -1 \Rightarrow \frac{d^2y}{dx^2} = 60x^3 - 30x = 60(-1)^3 - 30(-1) = -60 + 30 = -30 < 0$$

Hence, the function has a relative maximum value at $x = -1$.

$$\text{when } x = 1 \Rightarrow \frac{d^2y}{dx^2} = 60x^3 - 30x = 60(1)^3 - 30(1) = 60 - 30 = 30 > 0$$

Hence, the function has a relative minimum value at $x = 1$.

To find the points of inflection set $\frac{d^2y}{dx^2} = 0$ then solve for x .

$$\begin{aligned} \text{Thus we have: } \frac{d^2y}{dx^2} = 60x^3 - 30x = 0 &\Rightarrow 30x(2x^2 - 1) = 0 \\ &30x = 0; 2x^2 - 1 = 0 \\ &x = 0; x = \pm \sqrt{\frac{1}{2}} = \pm 0.71 \end{aligned}$$

Hence, the points of inflection are: $x = 0$; $x = -0.71$ and $x = 0.71$

We summarize the results in Table 12.3.3.4.3 on the next page.



Table 12.3.3.4.3

Interval	$x \leq -0.71$	$(-0.71, 0)$	$(0, 0.71)$	$x \geq 0.71$
Critical Numbers	-1			1
Point of Inflection		-0.71	0	0.71
Value of $f(x)$	2	1.5	0	-1.25
Value of $f'(x)$	0	-3.79	0	-3.79
Sign of $f'(x)$	-	0	0	+
Graph of $f(x)$	Concave downward			Concave upward
Conclusion	Relative maximum			Relative minimum

Sketch of the graph is shown below.

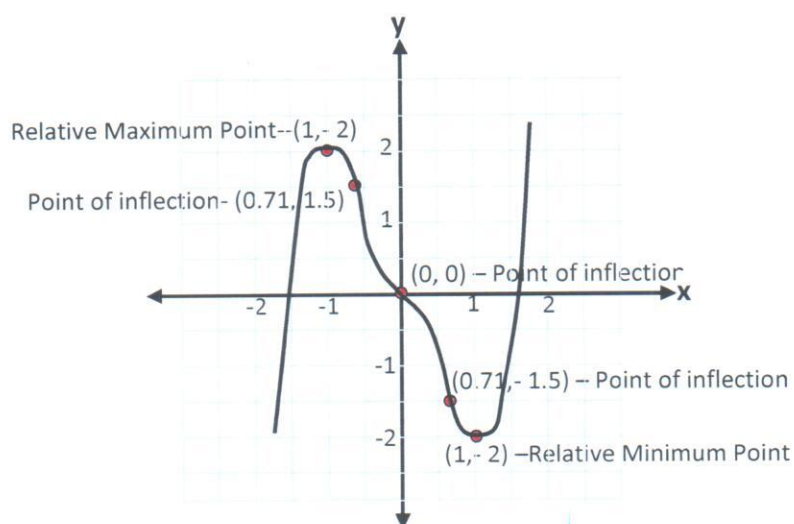


Figure 12.3.3.4.5: $y = 3x^5 - 5x^3$

Now do the learning activity on the next page.

**LEARNING ACTIVITY 12.3.3.4**

20 minutes

Solve the following problems.

1. Given the function $y = f(x) = x^3 + 3x^2 - 9x - 13$. Used the Second derivative to find:
 - a. Critical numbers
 - b. Relative maximum
 - c. Relative minimum
 - d. Sketch the graph

2. Given the function $y = f(x) = x^3 + 3x + 2$. Used the Second derivative test to find
 - a. Critical numbers
 - b. Relative maximum
 - c. Relative minimum
 - d. Points of inflection
 - e. Concavity
 - f. Sketch the graph of f .



12.3.3.5 Applied Maxima and Minima

In this section, we will show how the methods in the previous sections can be used to solve more applied maxima and minima optimization problems.

Strategy for solving Applied Maxima and Minima

1. Set up the equation whose maximum or minimum value is required. Make the necessary figures to make the solution clearer and easier.
2. If the resulting equation involves more than one variable, express these relations in terms of one variable using the sufficient conditions given in the problem.
3. apply the first and second derivatives tests.

Example 1

Find two positive real numbers x and y such that their sum is 100 and their product is as large as possible.

Solution: $x + y = 100$ Domain: $(0, 100)$
 $y = 100 - x$

Let P be the product of x and y , then $P = xy$

Substitute $y = 100 - x$ into $P = xy$ we have $P = x(100 - x)$

$$P = 100x - x^2$$

Solve for $\frac{dy}{dx}$: $P(x) = 100x - x^2 \Rightarrow \frac{dy}{dx} = 100x - x^2$
 $= 100 - 2x$

To find the critical points, set $\frac{dy}{dx} = 0$ and solve for x .

Hence, $100 - 2x = 0$
 $2x = 100$
 $x = 50$
and $y = 100 - x$
 $= 100 - 50$
 $= 50$

Find the second derivative $\frac{d^2y}{dx^2}$: $\frac{dy}{dx} = 100 - 2x \Rightarrow \frac{d^2y}{dx^2} = -2$

Since $\frac{d^2y}{dx^2}$ is always negative, therefore $x = 50$ is where the relative maximum occurs. Since $x = 50$ is the only critical number of the function on $(0, 100)$,



then $x = 50$ is where the function has its absolute maximum. Therefore, the product is maximum when $x = 50$ and $y = 50$.

The maximum product is 2500.

Example 2

The product of two positive numbers is 2304. Find the numbers so that their sum is a minimum.

Solution:

Let the numbers be x and y . $xy = 2304 \Rightarrow y = \frac{2304}{x}$

Let S be the sum, then $S = x + y$

$$S = x + \frac{2304}{x}$$

Differentiating $S'(x) = \frac{dy}{dx} = 1 - \frac{2304}{x^2}$ Domain: $(0, 2304)$

To find the critical points, set $\frac{dy}{dx} = 0$, and find x .

$$\begin{aligned} S'(x) = \frac{dy}{dx} = 1 - \frac{2304}{x^2} = 0 &\Rightarrow \frac{2304}{x^2} = 1 \\ x^2 &= 2304 \\ x &= \sqrt{2304} \\ x &= 48 \end{aligned}$$

Find the Second Derivative: $S'(x) = 1 - \frac{2304}{x^2} \Rightarrow S''(x) = \frac{d^2y}{dx^2} = \frac{4608}{x^3}$

When $x = 48$, $S''(48) = \frac{4608}{(48)^3} = \frac{1}{24}$

Since $S''(48) > 0$, S has a relative minimum at $x = 48$.

Since $x = 48$ is the only critical number of S on $(0, 2304)$, then S has an absolute minimum at $x = 48$.

Therefore, the sum is minimum at $x = 48$ and $y = 48$.



Example 3

A rectangle with a fixed perimeter of 64 units is rotated about one of its sides, thus sweeping out a figure in the shape of a right circular cylinder. What is the maximum volume of that cylinder?

Solution:

Let r be one side of the rectangle and h be the other side.

Therefore, $2r + 2h = 64$

$$h = 32 - r$$

Volume of the cylinder formed is:

$$V = \pi r^2 h$$

Substitute $h = 32 - r$ into the $V = \pi r^2 h$, we have $V = \pi r^2(32 - r)$

$$V = 32\pi r^2 - \pi r^3$$

Domain: $(0, 32)$

Differentiating $V = 32\pi r^2 - \pi r^3$ with respect to r we have, $\frac{dV}{dr} = 64\pi r - 3\pi r^2$

To find the critical point, set $\frac{dV}{dr} = 0$. Hence, $0 = 64\pi r - 3\pi r^2$

$$0 = \pi r(64 - 3r)$$

$$\pi r = 0; 64 - 3r = 0$$

$$r = 0; r = \frac{64}{3}$$

When $r = \frac{64}{3}$, $h = 32 - \frac{64}{3} = \frac{32}{3}$. Thus the height of the rectangle is $\frac{32}{3}$.

Find the second derivative: $\frac{dV}{dr} = 64\pi r - 3\pi r^2 \Rightarrow \frac{d^2V}{dr^2} = 64\pi - 6\pi r$

Substituting $r = \frac{64}{3}$ into $\frac{d^2V}{dr^2} = 64\pi - 6\pi r = 64\pi - 6\pi\left(\frac{64}{3}\right) = -64\pi$

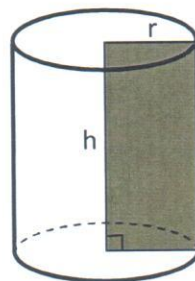
Since $\frac{d^2V}{dr^2}$ is always negative, then $r = \frac{64}{3}$ is a relative maximum.

When $r = 0$, $V = 32\pi(0)^2 - \pi(0)^3 = 0$

When $r = 32$, $V = 32\pi(32)^2 - \pi(32)^3 = 0$

Therefore, volume is maximum when $r = \frac{64}{3}$. The maximum volume is

$$V = \pi r^2(32 - r) = \pi \left(\frac{64}{3}\right)^2 \left(32 - \frac{64}{3}\right) = \frac{131072}{27} \pi \text{ cu. units.}$$





Example 4

A farmer is at a point on a riverbank. He wants to get to his cabin, located 3 km north and 8 km east. He can travel 5 km per hour along the river but only 3 km per hour over very rocky road. How far upriver should he go in order to reach the cabin in the shortest time?

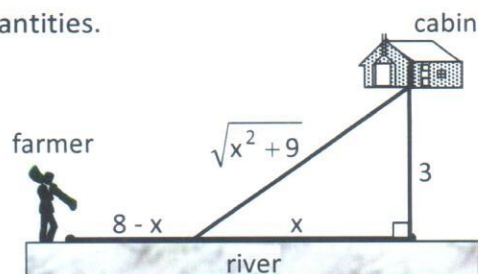
Solution:

We draw the figure and label the required quantities.

Let $8 - x$ = be the distance the farmer has to travel upriver

$\sqrt{x^2 + 9}$ = be the distance the farmer has to travel over rocky road

Domain: $[0, 8]$



We use the formula $\text{Time} = \frac{\text{Distance}}{\text{rate}}$ to express the relationship between the distance and the time.

Total Time = Time to travel upriver + Time to travel over rocky road

$$\begin{aligned} T_{\text{total}} &= \frac{D_{\text{river}}}{\text{rate}_{\text{river}}} + \frac{D_{\text{road}}}{\text{rate}_{\text{road}}} \\ &= \frac{8-x}{5} + \frac{\sqrt{x^2+9}}{3} \\ T &= \frac{8}{5} - \frac{x}{5} + \frac{1}{3}(x^2+9)^{\frac{1}{2}} \end{aligned}$$

Differentiating with respect to x : $\frac{dT}{dx} = -\frac{1}{5} + \frac{1}{6}(x^2+9)^{\frac{1}{2}}(2x) = -\frac{1}{5} + \frac{x}{3\sqrt{x^2+9}}$

To find the critical point, set $\frac{dT}{dx} = 0$ and solve for x .

$$0 = -\frac{1}{5} + \frac{x}{3\sqrt{x^2+9}}$$

$$\sqrt{x^2+9} = \frac{5x}{3}$$

$$x^2+9 = \frac{25x^2}{9}$$

$$9x^2+81=25x^2$$

$$16x^2=81$$

$$x^2 = \frac{81}{16}$$

$$x = \frac{9}{4} = 2\frac{1}{4} \text{ km}$$



Find the second derivative: $\frac{dT}{dx} = -\frac{1}{5} + \frac{x}{3\sqrt{x^2 + 9}} \Rightarrow T'(x) = \frac{d^2T}{dx^2} = \frac{3}{(x^2 + 9)^{\frac{3}{2}}}$

Evaluating $\frac{d^2T}{dx^2}$ when $x = \frac{9}{4}$:

$$T'\left(\frac{9}{4}\right) = \frac{3}{\left(\left(\frac{9}{4}\right)^2 + 9\right)^{\frac{3}{2}}}$$

$$= \frac{64}{1125}$$

Since $T'\left(\frac{9}{4}\right) > 0$, T is minimum when $x = \frac{9}{4}$.

Therefore, the farmer has to go $8 - x = 8 - \frac{9}{4} = \frac{23}{4} = 5\frac{3}{4}$ or 5.75 km upriver.

**LEARNING ACTIVITY 12.3.3.5**

20 minutes

Solve the following problems.

- The difference between two numbers is 18. Find the numbers so that their product is a minimum.

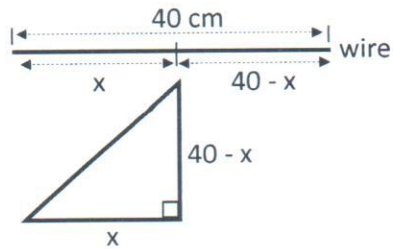
Answer: _____

- The sum of one number and three times the second number is 48. What numbers should be selected so that their product is as large as possible.

Answer: _____



3. A piece of wire 40-cm long is to be bent to form a right angle. Where should the vertex be to minimize the distance between the two loose ends?



Answer: _____



12.3.3.6 Differentials

We have used $\frac{dy}{dx}$ to denote the derivative of y with respect to x and treat $\frac{dy}{dx}$ as a single symbol not a ratio of dy and dx . In this section, we now define dy and dx . These symbols are called **differentials**.

The differential of x

If $y = f(x)$ is a differentiable function of x , then the **differential of x** , dx , is an increment of x , that is, $dx = \Delta x$. The differential of y is $dy = f'(x)dx$. Notice that in the equation $dy = f'(x)dx$ if both sides are divided by dx then

$$\frac{dy}{dx} = f'(x), \quad dx \neq 0$$

Thus $\frac{dy}{dx}$ can be thought of as the symbol for the first derivative of $f'(x)$, or as the quotient of differentials. Although it is true that $dx = \Delta x$, the differential dy is not the same as the increment in y . In general, $dy \neq \Delta y$.

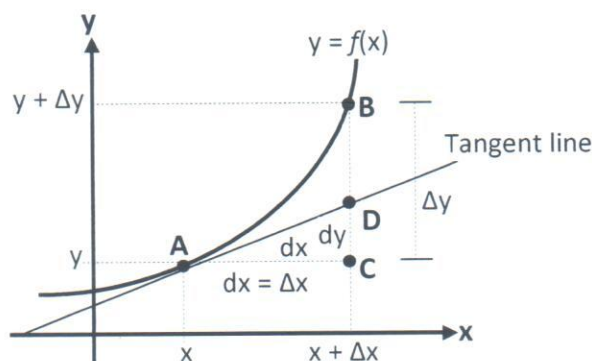


Figure 12.3.3.5.1: Differential

The tangent to the curve at point **A** is the line through points **A** and **D**. the derivative evaluated at point **A** gives the slope of the curve which is just the slope of the tangent line. But the ratio of the lengths of the line segments $\overline{DC}$ and $\overline{AC}$ is also the slope of the tangent line. (See Figure 12.3.3.5.1.) Thus,

$$f'(x) = \frac{\overline{DC}}{\overline{AC}}$$

But the length of $\overline{AC}$ is dx , therefore dy must be equal to the length of $\overline{DC}$, hence

$$f'(x) = \frac{\overline{DC}}{\overline{AC}} = \frac{\overline{DC}}{dx} \quad \Rightarrow \quad \overline{DC} = f'(x)dx = dy.$$



For any change, Δx in x , we obtain

$$\Delta y = f(x + \Delta x) - f(x)$$

This shown in the Figure 12.3.3.5.1 where the difference between Δy and dy is given by the line segment $\overline{BD}$.

We can use the dy as an approximation of Δy provided that Δy is sufficiently small.

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + dy$$

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

Note:

Although $dy \neq \Delta y$, it is a good estimate of Δy when dx is small. If dx were made very small then the difference between dy and Δy would also become very small.

Example 1

Find the differential of y for $y = f(x) = 3x^4 + 4x - 1$.

Solution: We use the formula $dy = f'(x)dx$.

Differentiating y with respect to x , we get

$$y = f(x) = 3x^4 + 4x - 1 \Rightarrow \frac{dy}{dx} = f'(x) = 12x^3 + 4$$

So that the differential dy can be written as $dy = f'(x)dx$.

Hence, $dy = (12x^3 + 4) dx$.

Example 2

Find dy for $y = f(x) = x^2 - 1$ when $x = 1$.

Solution: We use the formula $dy = f'(x)dx$

Differentiating y with respect to x , we get

$$y = f(x) = x^2 - 1 \Rightarrow \frac{dy}{dx} = f'(x) = 2x$$

Hence, $dy = (2x)dx$

When $x = 1$, $dy = [2(1)]dx$

Therefore, $dy = 2dx$.



Example 3

Find Δy for $y = x^2$.

Solution: The formula to find Δy is given by $\Delta y = f(x + \Delta x) - f(x)$

So, for $y = x^2$,

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= (x + \Delta x)^2 - x^2 \\ &= x^2 + 2x\Delta x + (\Delta x)^2 - x^2 \\ \Delta y &= 2x\Delta x + (\Delta x)^2\end{aligned}$$

Example 4

Find dy for $y = f(x) = x^3$ when $x = 2$ and $dx = 0.01$.

Solution: We use the formula $dy = f'(x)dx$

$$\begin{aligned}\text{So, for } y = f(x) = x^3 &\Rightarrow dy = (3x^2) dx \\ &= 3(2)^2(0.01) \\ &= 12(0.01) \\ dy &= 0.12\end{aligned}$$

Example 5

Find Δy and dy for $y = f(x) = x^2 + x$ when $x = 2$ and $\Delta x = 0.3$.

Solution: The formula to find Δy is given by $\Delta y = f(x + \Delta x) - f(x)$

$$\begin{aligned}\text{So, } \Delta y = f(x + \Delta x) - f(x) &\Rightarrow \Delta y = f(2 + 0.3) - f(2) \\ &= f(2.3) - f(2)\end{aligned}$$

$$\begin{aligned}\Delta y &= [(2.3)^2 + (2.3)] - [(2)^2 + (2)] \\ &= (5.29 + 2.3) - (4 + 2) \\ &= 7.59 - 6\end{aligned}$$

$$\Delta y = 1.59$$

To solve for dy , we use the formula $dy = f'(x)dx$.

Since $dx = \Delta x = 0.3$, and $f'(x) = 2x + 1$

$$\begin{aligned}dy &= f'(x)dx \\ dy &= (2x + 1)(0.3) \\ &= [2(2) + 1](0.3) \\ &= (5)(0.3)\end{aligned}$$

$$dy = 1.5$$

Therefore, $\Delta y = 1.59$ and $dy = 1.5$. They differ by **0.09**.

Example 6



Find the approximate volume of a wood required to make a box that can hold a cube 3 metres on a side. The box will be made using 1 cm thick wooden board.

Solution: The formula for the Volume of a Cube is given by $V = s^3$.

$$\text{Hence, } V = s^3 \Rightarrow dV = (3s^2)ds$$

Since $s = 3$ m and $\Delta s = ds = 1$ cm or 0.01 m

$$\begin{aligned}\text{Thus, } dV &= [3(3\text{m})^2][0.01\text{ m}] \\ &= (27\text{ m}^2)(0.01\text{ m}) \\ &= 0.27\text{ m}^3\end{aligned}$$

Therefore, the approximate volume of the required is 0.27 m^3 .

Example 7

Find an approximate value for $\sqrt{65}$ using $\Delta y = f'(x)\Delta x$

Solution: Since we are taking the square root of 65, we choose a perfect square number nearest to 65 and this is 64.

$$\begin{aligned}y = f(x) = \sqrt{x}, \text{ then } \Delta y &= \sqrt{x + \Delta x} - \sqrt{x} \\ &= \sqrt{x + \Delta x} - y\end{aligned}$$

$$\text{So, } y = \sqrt{x + \Delta x}$$

$$\text{Now, } \Delta y \approx f'(x)\Delta x \Rightarrow \Delta y = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}\Delta x$$

$$\text{Let } x = 64 \text{ and } \Delta x = 1, \text{ then } \Delta y = \frac{1}{2\sqrt{64}}(1)$$

$$\begin{aligned}&= \frac{1}{2(8)} \\ &= \frac{1}{16} = 0.0625\end{aligned}$$

$$\text{Therefore, } \sqrt{65} = \sqrt{64 + 1} \approx \sqrt{64} + 0.0625 = 8 + 0.0625 = \mathbf{8.0625}$$



Example 8

Find the approximate value for $\sqrt[4]{15}$.

Solution: Since we are taking the square root of 15, we choose a perfect square number nearest to 15 and this is 16.

$$y = f(x) = \sqrt[4]{x}, \text{ then } dy = f'(x) = \frac{dx}{4\sqrt[4]{x^3}}$$

$$\text{Now, } \Delta y \approx f'(x) \Delta x \Rightarrow \Delta y = \frac{dx}{4\sqrt[4]{x^3}}$$

$$\text{Let } x = 16 \text{ and } \Delta x = -1, \text{ then } \Delta y = 4\sqrt[4]{16} = 2$$

$$dy = \frac{-1}{4\sqrt[4]{(16)^3}} = -\frac{1}{4(\sqrt[4]{16})^3} = -\frac{1}{4(2)^3} = -\frac{1}{32} = -0.0325$$

$$\text{Therefore, } \sqrt[4]{15} = y + dy = 2 + (-0.0325) = \mathbf{1.9625}$$



LEARNING ACTIVITY 12.3.3.6



20 minutes

Solve the following problems.

1. Find dy for $y = x^2 + 7 + 1$.

Answer: _____

2. Find dy for $y = 2x^4 - 2x^3$

Answer: _____

3. Find dy and Δy for $y = 3x^2 + 400$, when $x = 100$ and $dx = 1$.

Answer: _____

4. Find an approximate value for $\sqrt{16.01}$ using $\Delta y = f'(x) \Delta x$.

Answer: _____

5. A spherical beach ball is being inflated. Find the approximate change in volume if the radius increases from 5 cm to 5.2 cm.

Answer: _____



40 minutes

A. Multiple Choice:

1. If $4x - 2y = 21$ and $\frac{dx}{dt} = 5$, then $\frac{dy}{dt}$ is

A. 8

B. 10

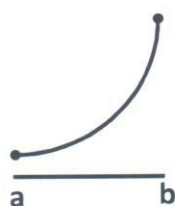
C. 15

D. 21

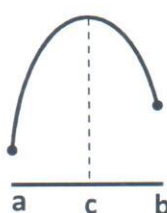
2. Which of the following figure shows the graph of a function which attains its absolute maximum value at the left of a of the closed interval $[a, b]$?



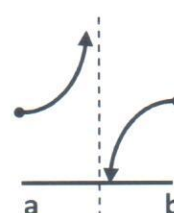
A



B



C



D

3. Find the minimum distance from the point $(4, 2)$ to the parabola $y^2 = 8x$.

A. $4\sqrt{3}$

B. $2\sqrt{2}$

C. $\sqrt{3}$

D. $2\sqrt{3}$

4. The sum of two positive numbers is 50. What are the numbers if their product is to be the largest possible.

A. 24 and 26

B. 28 and 22

C. 25 and 25

D. 20 and 30

5. Find the absolute minimum of the function $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.

A. -2

B. -3

C. 0

D. $\sqrt[3]{4}$

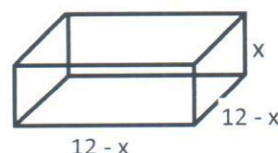
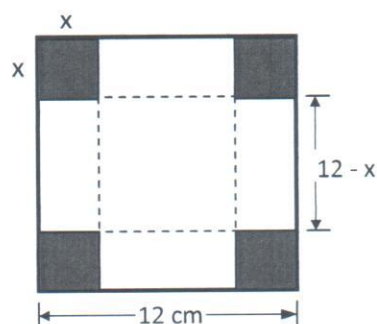


6. Find all the inflection points of the function $f(x) = 2x^3 + 3x^2 - 36x$.
- A. $x = 2, -3$ B. $x = \frac{1}{2}$
- C. $x = 0, \frac{1}{2}$ D. $x = 2$
7. At what value of x does $f(x) = (x - 1)^3(3 - x)$ have the absolute value maximum?
- A. 1 B. $\frac{3}{2}$
- C. 2 D. $\frac{5}{2}$
8. Find the critical point of the $g(x) = x^3 - 9x^2 + 15x - 7$
- A. $x = 1$ and $x = 5$ B. $x = 1$ and $x = \frac{3}{2}$
- C. $x = 1$ and $x = 2$ D. $x = 0$ and $x = \frac{5}{2}$
9. If $2x + y^2 = 9$ and $\frac{dy}{dt} = -6$, then the value of $\frac{dx}{dt}$ when $x = 4$ is
- A. ± 4 . B. ± 5 .
- C. ± 6 . D. ± 7 .
10. If $V = \pi r^2 h$ and $\frac{dV}{dt} = 4\pi$, $\frac{dr}{dt} = 3$, then the value of $\frac{dh}{dt}$ when $r = 1$ and $h = \frac{4}{3}$ is
- A. -1. B. -2.
- C. -3. D. -4.



B. Solve the following Problems.

1. An open square box is to be made by cutting congruent squares from the four corners of a square metal sheet and then folding up the resulting flaps (see the figure below). If the length of a side of the metal sheet is 12 cm, how should the square be cut in order to make a box with the largest possible volume?



Answer: _____

2. Construct a table that indicates the intervals where the function $f(x) = \frac{1}{2}(x-1)^2(4-x)$ is increasing and/or decreasing and where the graph of the function is concave upward or downward. Also indicate the relative extrema of the function and the points of inflection of the graph of the function, if there is/are any.

Answer:

Interval					
x					
Value of $f(x)$					
Value of $f'(x)$					
Conclusion					



12.3.4: INTEGRAL CALCULUS

We have shown the different methods for finding the derivatives of algebraic functions in the previous sections. We also studied some applications of the derivative.

In this section we will turn our attention to reversing the operation of differentiation. That is finding the function when the derivative is given. This process is called **antidifferentiation** or **integration**.

Definition

An **antiderivative** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ for all x .

Example 1

If $F(x) = 16x$, then $F'(x) = 16$, so $F(x) = 16x$ is an antiderivative of $f(x) = 16$.

Example 2

If $F(x) = 12x^2$, then $F'(x) = 24x$, making $F(x) = 12x^2$ an antiderivative of $f(x) = 24x$.

Example 3

Find an antiderivative of $f(x) = 4x^7$.

Solution: To find a function $F(x)$ whose derivative is $4x^7$, work backwards. We know that the derivative of ax^n is nax^{n-1} . So, if

$$nax^{n-1} \text{ is } 4x^7, \text{ then } n - 1 = 7 \Rightarrow n = 8$$

and

$$na = 4 \Rightarrow a = \frac{4}{n} \Rightarrow a = \frac{4}{8} = \frac{1}{2}$$

Therefore, $\frac{1}{2}x^8$ is an antiderivative of $4x^7$.

Consider the function $F(x) = x^3$, is not only the function whose derivative is $f(x) = 3x^2$. The function $G(x) = (x^3 + 1)$ and $H(x) = (x^3 - 4)$ also have $f(x) = 3x^2$ as a derivative. In fact for any real number C , $F(x) = x^3 + C$ has $f(x) = 3x^2$ as derivative. This shows that there is a family or class of functions having $3x^2$ as derivative. We shall use of the fact that if two functions $F(x)$ and $G(x)$ have the same derivative for all values of x in the interval $a < x < b$, then $F(x)$ and $G(x)$ differ by a constant, that is $F(x) - G(x) = C$.

If $F(x)$ and $G(x)$ are both antiderivative of a function $f(x)$, then there is a constant C such that

$$F(x) - G(x) = C$$

(Two antiderivatives of a function can differ only by a constant.)

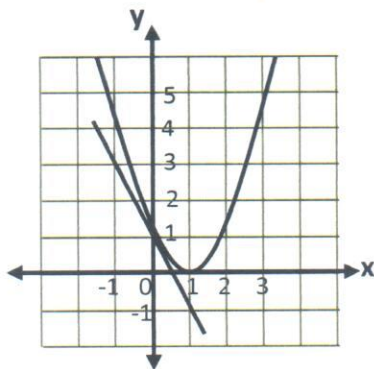


For example

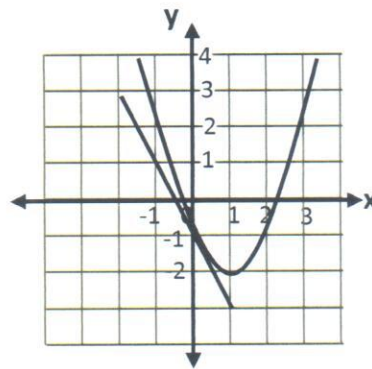
$$F(x) = x^2 - 2x + 1, \quad G(x) = x^2 - 2x - 1, \quad H(x) = x^2 - 2x + 2$$

are all antiderivatives of $f(x) = 2x - 2$ and any of them differ only by a constant.

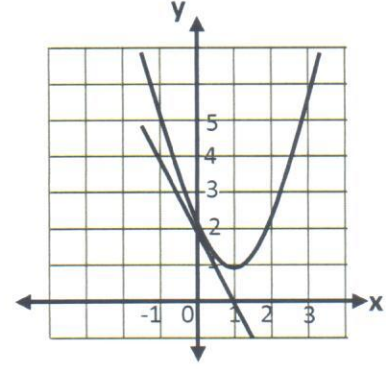
The derivative of a function gives the slope of a tangent line at any value of x . The fact that these three functions have the same derivative, $f(x) = 2x - 2$, means that their slopes at any particular value of x (in this case at $x = 0$) are the same as shown in the figure below.



$$F(x) = x^2 - 2x + 1$$



$$G(x) = x^2 - 2x - 1$$



$$H(x) = x^2 - 2x + 2$$

12.3.4.1 The Indefinite Integral

The family of all antiderivatives of the function f is denoted by $\int f(x)dx$.

That is, if $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C \quad \text{for any real numbers } C.$$

This process of evaluating $\int f(x)dx$, that is, finding $F(x) + C$ is called **integration**. The arbitrary constant C is called the **constant of integration**. The symbol $\int$ is the **integral sign** which means sum devised by Leibniz. The function $f(x)$ is the **integrand of the integral**, and $\int f(x)dx$ is called an **indefinite integral**. The dx refers to the idea that the function $f(x)$ is to be antidifferentiated or integrated with respect to the variable x . Thus

$$\int f(x)dx = F(x) + C \quad (1)$$

Simply means that

$$F'(x) = f(x) \quad (2)$$

Giving us

$$\frac{d}{dx} [F(x) + C] = f(x) \Rightarrow \frac{d}{dx} [F(x)] = f(x) \quad (3)$$

Evaluating an indefinite integral, say $\int f(x)dx$ is equivalent to finding $F(x) + C$ such that F satisfies (1) and (2).



Example 1

Verify that $\int (2x + 3)dx = x^2 + 3x + C$.

Solution: Let $F(x) = x^2 + 3x$ and $f(x) = 2x + 3$.
Since $F'(x) = 2x + 3 = f(x)$, it follows from **(1)** and **(2)**
That $\int (2x + 3)dx = x^2 + 3x + C$

Example 2

Express the statement $\int (2x + 3)dx = x^2 + 3x + C$.

Using the $\frac{d}{dx} [F(x)] = f(x)$: $\int (2x + 3)dx = x^2 + 3x + C \Rightarrow d(x^2 + 3) = (2x + 3)dx$.

Four Basic Rules of Differentiation

1. **The Power Rule:** For any real number $n \neq -1$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

In words, when x^n is integrated, the exponent n and x is increased by 1 and then x^{n+1} is divided by the new exponent $n + 1$.

2. **The Constant Multiple Rule:** If $\int f(x)dx$ exists, then

$$\int cf(x)dx = c \int f(x)dx, \text{ where } c \text{ is any constant.}$$

In words, the integral of a constant times a function equals the constant times the integral of the function.

3. **The Sum or difference Rule:** If $\int f(x)dx$ and $\int g(x)dx$ exist, then

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

In words, the integral of a sum or difference of two functions is the sum or difference of the integrals.



Example 1 Evaluate the following integrals.

a. $\int dx$

Solution:

$$\begin{aligned}\int dx &= \int x^0 dx = \frac{x^{0+1}}{0+1} + C \\ &= \frac{x^1}{1} + C \\ &= x + C\end{aligned}$$

b. $\int 2x dx$

Solution:

$$\begin{aligned}\int 2x dx &= 2 \int x dx = 2 \left(\frac{x^{1+1}}{1+1} \right) + C \\ &= 2 \left(\frac{x^2}{2} \right) + C \\ &= x^2 + C\end{aligned}$$

Example 2 Evaluate the following integrals:

a. $\int (2x^2 + 3x - 4) dx$

Solution: Applying the rules for integration, we get

$$\begin{aligned}\int (2x^2 + 3x - 4) dx &= 2 \int x^2 dx + 3 \int x dx - 4 \int dx \\ &= 2 \left(\frac{x^3}{3} \right) + 3 \left(\frac{x^2}{2} \right) - 4x + C \\ &= \frac{2}{3} x^3 + \frac{3}{2} x^2 - 4x + C\end{aligned}$$

$$\text{Thus, } \int (2x^2 + 3x - 4) dx = \frac{2}{3} x^3 + \frac{3}{2} x^2 - 4x + C$$

b. $\int (\sqrt{x} + \sqrt[3]{x}) dx$

Solution: Applying the rules for integration, we get

$$\begin{aligned}\int (\sqrt{x} + \sqrt[3]{x}) dx &= \int \sqrt{x} dx + \int \sqrt[3]{x} dx \\ &= \int x^{\frac{1}{2}} dx + \int x^{\frac{1}{3}} dx \\ &= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C\end{aligned}$$

$$\text{Thus } \int (\sqrt{x} + \sqrt[3]{x}) dx = \frac{2}{3} x^{\frac{3}{2}} + \frac{3}{4} x^{\frac{4}{3}} + C$$



4. **The General Power Rule:** If $u = f(x)$ and $du = f'(x) dx$ exists, then

$$\int [f(x)]^n [f'(x)] dx = \frac{[f(x)]^{n+1}}{n+1} + C, \text{ where } n \neq -1$$

Or using the differentials, we get

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \text{ where } n \neq -1$$

Example 3 Evaluate the following integrals using two methods:

a. $\int (2x + 1)^2 dx$

b. $\int (x - 1)^3 dx$

Solution: In Method 1, we will expand first the integrand before we apply the Power Rule.
In Method 2, we will use the General Power Rule.

Method 1(Expanding the integrand)

$$\begin{aligned} \int (2x + 1)^2 dx &= \int (4x^2 + 4x + 1) dx \\ &= 4 \left(\frac{x^{2+1}}{2+1} \right) + 4 \left(\frac{x^{1+1}}{1+1} \right) + x + C \\ &= \frac{4}{3} x^3 + 2x^2 + x + C \end{aligned}$$

Method 2(Using the General Power Rule)

We let $f(x) = 2x + 1$, so that $f'(x) = 2dx$. Then, we introduce 2 under the integral sign with its reciprocal in front. Hence,

$$\begin{aligned} \int (2x + 1)^2 dx &= \frac{1}{2} \int (2x + 1)^2 (2dx) \\ &= \frac{1}{2} \left(\frac{(2x + 1)^{2+1}}{2+1} \right) + C \\ &= \frac{1}{6} (2x + 1)^3 + C \end{aligned}$$



The Velocity and Acceleration

We will now show how the techniques of differentiation and integration can be used to study various types of motion.

Example 1

Suppose that the velocity of a car at time t is $v(t) = 3t + 5$. At $t = 0$, the car starts from position O in the figure below. Find the distance function $s(t)$.



Solution: Integrating the velocity function, we get

$$s(t) = \int v(t) dt \Rightarrow s(t) = \int (3t + 5) dt \Rightarrow s(t) = \frac{3}{2}t^2 + 5t + C$$

$$\text{At } t = 0, \text{ we see that } s(0) = 0 \Rightarrow 0 = \frac{3}{2}(0) + 5(0) + C \Rightarrow C = 0$$

$$\text{Hence, } s(t) = \frac{3}{2}t^2 + 5t$$

Example 2

The manufacturer of a certain type of automobile advertises that it will accelerate from 0 to 100 km per hour in 1 minute. If its acceleration is constant, how far will the car go in this length of time?

Solution: Let $s(t)$ – be the distance function relative to the starting point
 $v(t)$ – be the velocity at any time t
 $a(t)$ – be the acceleration at any time t

$$t = 1 \text{ minute} \Rightarrow t = \frac{1}{60} \text{ hours} \Rightarrow v(t) = v\left(\frac{1}{60}\right) = 100 \text{ kph}$$

At $t = 0$, $s(0) = 0$, $v(0) = 0$, and $a(t) = k$ for some constant k . Notice that

$$\frac{dv(t)}{dt} = a(t) = k$$

$$v(t) = \int a(t) dt \Rightarrow v(t) = \int k dt \Rightarrow v(t) = kt + C_1$$

Since $v(0) = 0 \Rightarrow 0 = k(0) + C_1$, so that $C_1 = 0$ and $v(t) = kt$. Moreover, since



$$v\left(\frac{1}{60}\right) = 100 \Rightarrow 100 = k\left(\frac{1}{60}\right) \Rightarrow k = 6000$$

Now, $v(t)$ becomes $v(t) = 6000t$. Integrating $v(t)$ with respect to t , we get the distance function $s(t)$. Thus,

$$s(t) = \int v(t)dt \Rightarrow s(t) = \int (6000t)dt \Rightarrow s(t) = 3000t^2 + C \text{ car}$$

Since, $s(0) = 0$, we see that $3000(0)^2 + C_2 = 0 \Rightarrow C_2 = 0$.

$$\text{Thus, } s(t) = 3000t^2 \text{ and } s\left(\frac{1}{60}\right)^2 = 3000\left(\frac{1}{60}\right)^2 = 0.833$$

Therefore, the car travels 0.833 km in 1 minute.

Now do the learning activity 12.3.4.1.



LEARNING ACTIVITY 12.3.4.1



20 minutes

A. Evaluate the following integrals.

1. $\int (2x + 1)dx$

Answer: _____

2. $\int (x^2 + 3x + 2)dx$

Answer: _____

3. $\int (x^3 - x)dx$

Answer: _____

4. $\int (4x - 3)^3 dx$

Answer: _____



12.3.4.2 The Definite Integrals

In this section, we will define the definite integral, discuss the Fundamental Theorem of calculus, apply its properties to evaluate definite integrals and find the area the curve and area between two curves.

The Definite Integral

Suppose you know the rate $f(x) = \frac{dF}{dx}$ at which a certain quantity is changing and wish to find the amount by which the quantity F will change between $x = a$ and $x = b$. you would first find F by antidifferentiation and then compute the difference

$$\text{Change in } F \text{ between } x = a \text{ and } x = b = F(b) - F(a)$$

The numerical result of such a computation is called a **definite integral** of the function f and is denoted by the symbol $\int_a^b f(x)dx$.

The definite integral of f from a to b is denoted by

$$\int_a^b f(x)dx = F(x) \Big|_a^b = F(b) - F(a)$$

where F is an antiderivative of f . That is, the definite integral is the net change in the antiderivative between $x = a$ and $x = b$.

The symbol $\int_a^b f(x)dx$ is read as “the (definite) integral of f from a to b ”.

This notation suggests that we first find an antiderivative $F(x)$ of $f(x)$. Then write $F(x) \Big|_a^b$ as an aid in computing $F(b) - F(a)$. Specifically, we replace x by the upper limit b , obtaining $F(b)$, and from this subtract the value $F(a)$ by setting $x = a$.

The Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function on $[a, b]$ and let $F(x)$ be any antiderivative for $f(x)$. That is, let $F'(x) = f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The Fundamental Theorem of calculus does not define the definite integral. It merely provides a method for evaluating the definite integral.

Example



Evaluate the following definite integrals.

a. $\int_0^2 x dx$

b. $\int_1^3 (x+1)^2 dx$

c. $\int_{-1}^1 (4x^3 - 5x^4) dx$

Solution:

a. $\int_0^2 x dx = \frac{1}{2} x^2 \Big|_0^2 = \frac{1}{2} (2) - \frac{1}{2} (0) = 1 - 0 = 1$

b. $\int_1^3 (x+1)^2 dx = \frac{1}{2} (x+1)^2 \Big|_1^3 = \frac{1}{2} (3+1)^2 - \frac{1}{2} (1+1)^2 = \frac{1}{2} (4)^2 - \frac{1}{2} (2)^2 = 8 - 2 = 6.$

c. $\int_{-1}^1 (4x^3 - 5x^4) dx = \frac{1}{4} (4x^4) - \frac{1}{5} (5x^5) = \left[x^4 - x^5 \right]_{-1}^1$
 $= [(1)^4 - (1)^5] - [(-1)^4 - (-1)^5]$
 $= [1 - 1] - [1 + 1]$
 $= 0 - 2$
 $= -2$

Properties of Definite Integrals

The following properties are possessed by all definite integrals.

1. $\int_a^b f(x) dx = 0$

In words, the definite integral over an interval of length zero must equal zero.

2. $\int_a^b f(x) dx = - \int_b^a f(x) dx$

In words, interchanging the limits changes the sign of the integral.

3. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

In words, the interval of integration may be broken up into any number of subintervals, and the integration is performed over each interval separately.

4. $\int_a^b f(x) dx = \int_a^b f(z) dz$

In words, it makes no difference what letter is used for the variable of integration. That is, the definite integral of a given integrand is independent of the variable of integration.



LEARNING ACTIVITY 12.3.4.2



20 minutes

Evaluate the following integrals.

1. $\int_0^4 (x^3 - x^2 + 1) dx$

Answer: _____

2. $\int_{-3}^5 (x^3 - 4x) dx$

Answer: _____

3. $\int_{-1}^3 (3x^2 + 5x - 1) dx$

Answer: _____

4. $\int_1^3 (x^3 - x) dx$

Answer: _____

5. $\int_3^3 (x^3 - x) dx$

Answer: _____



12.3.4.3 Area under the Curve

If $f(x)$ is a continuous function on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$ then the area of the region bounded by the graph of $y = f(x)$ (see figure 12.3.4.1a) and the x -axis between $x = a$ and $x = b$ is given by the definite integral below.

$$\text{Area} = \int_a^b f(x) dx$$

Notice that the area is approximated by the areas of the rectangles whose heights are $f(x_i)$ and whose widths are dx . (See figure 12.3.4.1b).

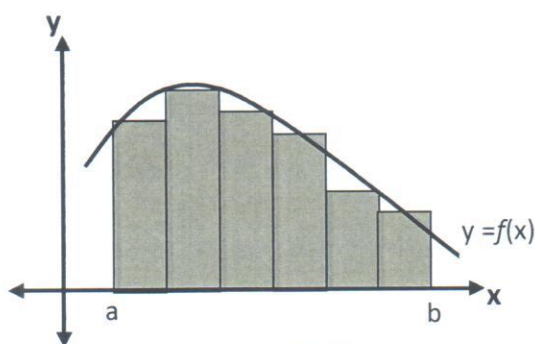


Figure 12.3.4.1a

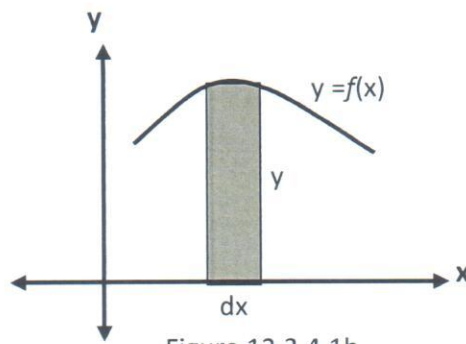


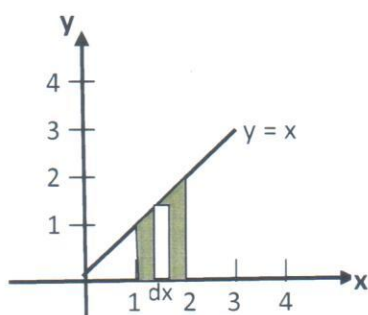
Figure 12.3.4.1b

Area under the Curve

Example 1

Find the area bounded by the line $y = x$ from $x = 1$ to $x = 2$.

Solution: The region whose area we must calculate is the shaded region in figure below.



Region bounded by $y = x$, $x = 1$, and $x = 2$.

Here $y = x$, with lower and upper limits $a = 1$ and $b = 2$, respectively. The area of the shaded region can be computed using the area of vertical rectangular strip. Hence,

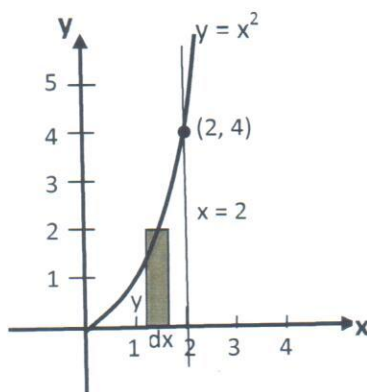
$$A = \int_1^2 y dx \Rightarrow A = \int_1^2 x dx \Rightarrow A = \left. \frac{1}{2} x^2 \right|_1^2 = \left[\frac{1}{2} (2)^2 \right] - \left[\frac{1}{2} (1)^2 \right] = \frac{3}{2} \text{ sq. units}$$



Example 2

Find the area under the parabola $y = x^2$ from $x = 0$ and $x = 2$.

Solution: The area of the region we want to calculate is shown in the graph below.



The area of the shaded rectangle is $A = LW$, where $L = y$ and $W = dx$. Since x is the variable of integration, so $x \in [0, 2]$. The lower and upper limits are $a = 0$, and $b = 2$, respectively.

Hence,

$$dA = \int_0^2 x dx, x \in [0, 2] \Rightarrow A = \int dA \Rightarrow A = \int_0^2 y dx$$

$$A = \int_0^2 x^2 dx \Rightarrow A = \frac{1}{2} x^3 \Big|_0^2 = \left[\frac{1}{3} (2)^3 \right] - \left[\frac{1}{3} (0)^3 \right]$$

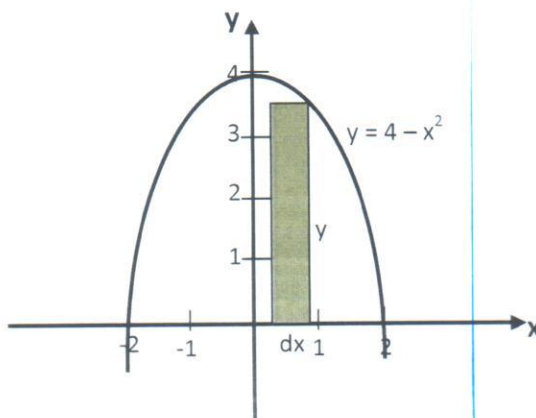
$$A = \frac{8}{3} - \frac{0}{3}$$

$$A = \frac{8}{3} \text{ square units}$$

Example 3

Find the area bounded by the curve $y = 4 - x^2$ and the x -axis

Solution: The area of the region we want to calculate is shown in the graph below.





The differential equation for the area of the shaded rectangle is

$$dA = ydx, x \in [-2, 2]$$

The area of the region is $A = \int dA \Rightarrow A = \int_{-2}^2 ydx \Rightarrow \int_{-2}^2 (4 - x^2)dx$

$$\begin{aligned} A &= 4x - \frac{1}{3}x^3 \Big|_{-2}^2 = \left[4(2) - \frac{1}{3}(2)^3 \right] - \left[4(-2) - \frac{1}{3}(-2)^3 \right] \\ &= \left[8 - \frac{8}{3} \right] - \left[-8 + \frac{8}{3} \right] \\ &= 16 - \frac{16}{3} \\ &= \frac{32}{3} \text{ sq. units} \end{aligned}$$

Another alternative solution: The shaded region we will consider here is horizontal and the variable of integration is y . (see figure below).

Solving the given function $y = 4 - x^2$ for y in terms of x , we get $y = 4 - x^2 \Rightarrow x = \sqrt{4 - y}$.

We calculate the area of the region in the following way:

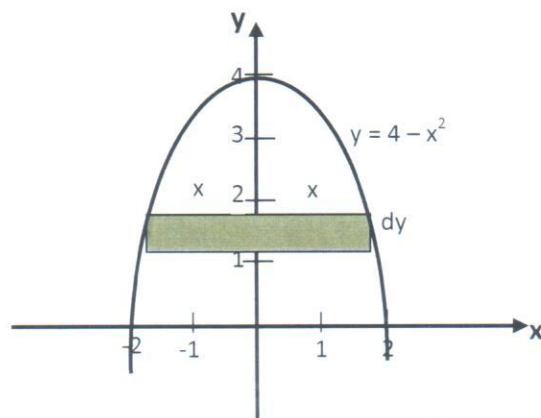
$$dA = 2xdy, \quad y \in [0, 4] \Rightarrow A = \int dA \Rightarrow A = \int_0^4 2xdy$$

$$A = \int_0^4 2\sqrt{4 - y} dy \Rightarrow A = 2 \int_0^4 (4 - y)^{\frac{1}{2}} dy$$

Let $u = 4 - y$ so that $du = -dy$.

$$A = -2 \int_0^4 (4 - y)^{\frac{1}{2}} dy \Rightarrow A = -\frac{4}{3} (4 - y)^{\frac{3}{2}} \Big|_0^4$$

$$A = \left[-\frac{4}{3} (4 - 4)^{\frac{3}{2}} \right] - \left[-\frac{4}{3} (4 - 0)^{\frac{3}{2}} \right] = 0 + \frac{4}{3} (4)^{\frac{3}{2}} = \frac{32}{3} \text{ sq. units}$$



Region bounded by the curve $y = 4 - x^2$ and the x -axis



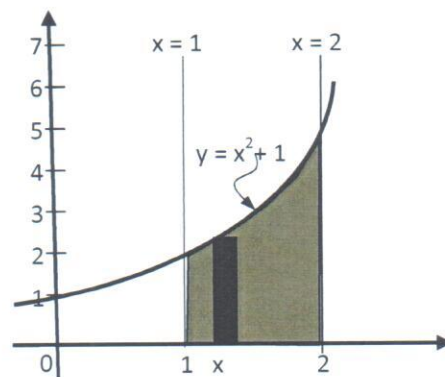
LEARNING ACTIVITY 12.3.4.3



20 minutes

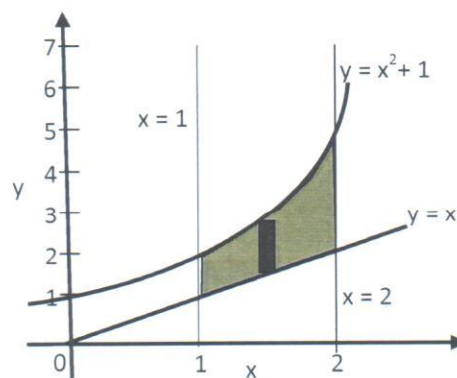
Solve the following:

1. Find the area of the region bounded by the graphs of $y = x^2 + 1$, $x = 1$, $x = 2$ and $y = 0$.



Answer: _____

2. Find the area of the region bounded by the graphs of $y = x^2 + 1$, $x = 1$, $x = 2$ and $y = x$.



Answer: _____

**SUMMATIVE TASK 12.3.4**

40 minutes

A. Multiple Choice

1. The process of finding a function when the derivative is given is called

- | | |
|--------------------|------------------|
| A. differentiation | B. extrapolation |
| C. integration | D. interpolation |

Refer to the formula $\int f(x)dx = F(x) + C$,

2. $F(x) + C$ is called the

- | | |
|------------------------|----------------------------|
| A. integrand | B. definite integral |
| C. indefinite integral | D. constant of integration |

3. C is called the

- | | |
|------------------------|----------------------------|
| A. integrand | B. definite integral |
| C. indefinite integral | D. constant of integration |

4. Evaluate $\int 4dx$

- | | |
|----------------|----------------------|
| A. $4x + C$ | B. $\frac{x}{4} + C$ |
| C. $(4+x) + C$ | D. 4 |

5. $\int_0^8 \frac{dx}{\sqrt{x+1}} =$

- | | |
|------|------------------|
| A. 1 | B. $\frac{3}{2}$ |
| C. 2 | D. 4 |



B. Evaluate the given integral.

1. $\int (2x + 3)dx$

Answer: _____

2. $\int (x^2 + 3x + 2)dx$

Answer: _____

3. $\int (2x + 3)(x^2 + 3x + 2)^3 dx$

Answer: _____

4. $\int_1^3 (3x + 2)dx$

Answer: _____

C. Solve the following problem.

1. Find the area of the region bounded by graphs of $y = x^2 - 2$ and $y = x$.
Sketch the graph.

Answer: _____

END OF UNIT MODULE 12.3



UNIT SUMMARY

This summary the important key ideas and concepts to be remembered.

12.3.1 Limits and Continuity

- A **function** is a special type of a relation such that no two ordered pairs (x, y) of the set have different second coordinates corresponding to the same first coordinate.
- **Domain** is the first coordinates (x) of the ordered pairs.
- **Range** is the second coordinates (y) of the ordered pairs.
- A function is denoted by the symbols $y = f(x)$ which is read as “ y is a function of x ” where y is called the **value of the function**.
- To **evaluate a function** for a particular value of x , replace x by that value within each set of parentheses.
- Different Types of Function
 - **Linear Function** – simplest type of function, where the graph is a **straight line**.
 - Any equations of the form $y = mx + b$ are called linear functions.
 - If $m = 0$, then $f(x) = b$ is called a **constant function**.
 - If the domain of a constant function is a set of real numbers, then the graph of $f(x) = b$ is a **horizontal line**.
 - The line is parallel to the y -axis is a **vertical line**. Note that vertical lines are not functions.
 - **Quadratic Function** – any equation in the form of $y = ax^2 + bx + C$, where $a \neq 0$. The graph is a **parabola**.
 - If $a > 0$, the parabola opens upward and has a minimum value at the vertex.
 - If $a < 0$, the parabola opens downward and has a maximum value at the vertex.
 - **Cubic Function** – any equation in the form of $y = ax^3 + bx^2 + cx + a$, where $a \neq 0$. The graph of a cubic function is called a **cubic curve**. It looks like two halves of a parabola in opposite directions.
- **Limit** is used in mathematics to describe the behaviour of that **function** $f(x)$ as it approaches a specific value of x .
 - The notation $\lim_{x \rightarrow a} f(x) = L$, means that as x gets closer to a but not equal to a , $f(x)$ gets closer to L .
 - The limit of a function as x approaches a is independent of the value of the function at a even though $\lim_{x \rightarrow a} f(x)$ exists, the value of the function at a maybe:
 - undefined
 - the same as the limit
 - defined but different from the limit
- **Properties of Limits**
 - If $f(x)$ is continuous at $x = a$, then $\lim_{x \rightarrow a} f(x) = f(a)$
 - The limit of a constant is the constant itself.



- The limit of a sum or difference equals the sum and difference of the limits.
- The limit of a product is equal to the product of the limits.
- The limit of a quotient is equal to the quotient of the limits.
- Left-hand limit- is the limit of a function $f(x)$ at a point a as x approaches a from decreasing values of a .
- Right-hand limit- is the limit of a function $f(x)$ at a point a as x approaches a from increasing values of a .
- When the left-hand limit is equal to the right-hand limit we say the limit exists thus we say that the function has a **two sided limit**.
- A function is said to be continuous at a point $x = a$ if the following conditions are satisfied:
 - $f(x)$ is defined, that is, exists, at $x = a$
 - $\lim_{x \rightarrow a} f(x)$ exists
 - $\lim_{x \rightarrow a} f(x) = f(a)$
- A function is said to be continuous at an interval (a, b) if it is continuous at each point in that interval
- A function is said to have a removable continuity if $\lim_{x \rightarrow 0} f(x)$ exists $\lim_{x \rightarrow 0} f(x) = f(a)$. is the
- A function is said to have jump discontinuity at $x = a$, if $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ for any positive integer.

12.3.2: the Derivatives

- The **Slope of a Curve** $y = f(x)$ at $[x, f(x)]$ is equal to the slope of its tangent line at $[x, f(x)]$. It is defined as the **first derivative** of the function $y = f(x)$, defined by

$$m = \frac{dy}{dx} = \lim_{x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- **Differentiation** process of obtaining the derivative of a function.
- Finding the slope of a tangent by evaluating the limit of the slope of a secant is known as **differentiation from first principle**.
- **Rules for differentiation**
 - The Derivative of a constant is zero.
 - The derivative of the n^{th} power of a variable is the product of n and the $(n-1)^{\text{st}}$ power of the variable.
 - A Constant factor can be moved through a derivative sign.
 - The derivative of a sum or difference of a series of terms is equal to the sum and difference of the derivatives of each individual term.
 - The derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.
 - The derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.
- **The Chain Rule-** If y is a function of u and u is a function of x , then y is a function of x or a composite function. Hence, The derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x .



- **The General Power Rule-** The derivative of the n^{th} power of a differentiable function is the product of n , $(n - 1)$ st power of the function, and the derivative of the function.
- **Higher Order Derivatives** are the result of two or more successive differentiations of a function.
- **Average Rate of Change** of $y = f(x)$ with respect to x in an interval $[x, x + \Delta x]$ is the slope of the secant line joining the points, $(x, f(x))$ and $(x + \Delta x, f(x) + \Delta x)$ on the graph of $y = f(x)$.
- **Instantaneous Rate of Change** of $y = f(x)$ with respect to x at the point $(x, f(x))$ is the slope of the tangent line to the graph of $y = f(x)$.
- **Rectilinear Motion** is any motion along a straight line. **Positive distance** may be interpreted as motion to the right (or upward). **Negative distance** may be interpreted as motion to the left (or downward).
- The **Average velocity** Δv of a particle at time t_1 to t_2 is $\Delta v = \frac{\Delta s}{\Delta t}$
- The **Instantaneous velocity** $v(t)$ of a particle at time t is $v(t) = s'(t) = \frac{ds}{dt}$
- The **speed** of a particle at time t is $|v(t)| = \left| \frac{ds}{dt} \right|$
- The **acceleration** $a(t)$ of a particle at time t is $a(t) = v'(t) = \frac{dv}{dt}$
- **The Derivative of Trigonometric Functions**
 - The derivative of $\sin x$: $\frac{d}{dx}(\sin x) = \cos x$
 - The derivative of $\cos x$: $\frac{d}{dx}(\cos x) = -\sin x$
 - The derivative of $\tan x$: $\frac{d}{dx}(\tan x) = \sec^2 x$
 - The derivative of $\csc x$: $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
 - The derivative of $\sec x$: $\frac{d}{dx}(\sec x) = \sec x(\tan x)$
 - The derivative of $\cot x$: $\frac{d}{dx}(\cot x) = -\csc^2 x$

12.3.3: Curve Sketching

- **Relative extrema** refers to the highest or lowest values in the graph of a function. The highest value is called **relative maxima** and the lowest value is called **relative minima**. Sometimes these points are called the **local maximum** or **local minimum**.
- The **critical number** of a function is any value of x in the domain of the function at which the function is equal to zero or at which the function is differentiable.
- If a graph of a function rises from left to right over an interval, it is said to be an **increasing function**.
- If a graph of a function drops from left to right over an interval, it is said to be a **decreasing function**.
- Derivatives are used to determine whether a function is increasing or decreasing.



- If $\frac{dy}{dx}$ is positive at a point on a curve, then the tangent is positive at that point and the function is said to be an **increasing function** at that point.
- If $\frac{dy}{dx}$ is negative at a point on a curve, then the tangent is negative at that point and the function is said to be an **decreasing function** at that point.
- If $\frac{dy}{dx} = 0$ at a point on a curve, then the tangent is parallel to the x-axis at that point and the graph is said to have a **stationary point** at that point.
- **Point of inflection** is a point where the graph crosses the tangent line and occurs where the concavity of the graph changes.
- **The Differentials** – If $y = f(x)$ is a differentiable function of x , then the **differential of x** , dx is an increment of x , that is $dx = \Delta x$. The **differential of y** is $dy = f'(x)dx$.
- $\frac{dy}{dx}$ is the symbol for the first derivative of $f'(x)$, or as the quotient of differentials.

12.3.4: Integral Calculus

- **Integration** or **Antidifferentiation** is the process of finding the function when the derivative is given.
- The **antiderivative** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ for all x .
- The family of antiderivatives of the function f is denoted by the symbol $\int f(x)dx = F(x) + C$, where C is called the **constant of integration**, $\int$ is the **integral sign** meaning **sum**, $f(x)$ is the **integrand** of the integral, and $f(x)dx$ is called the **indefinite integral**.
- The fundamental Theorem of Calculus is denoted by $\int_a^b f(x)dx = F(b) - F(a)$
- **Properties of Definite Integrals**
 - $\int_a^b f(x)dx = 0$
 - $\int_a^b f(x)dx = \int_b^a f(x)dx$
 - $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
 - $\int_a^b f(x)dx = \int_a^b f(z)dz$
- **The Area Under the Curve** is the region bounded by the graph $y = f(x)$ and the x-axis between $x = a$ and $x = b$ and is denoted by the formula: $\text{Area} = \int_a^b f(x)dx$

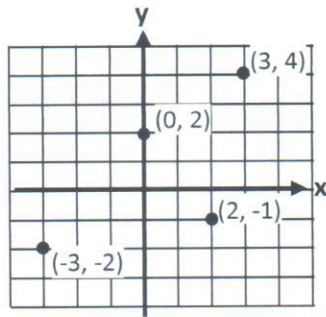
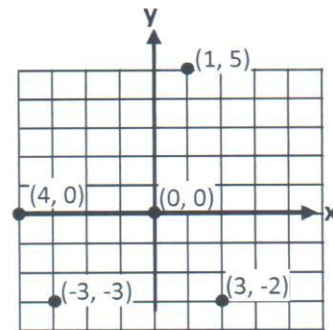
REVISE SECTIONS 12.3.1 TO 12.3.4 THEN DO MODULE ASSESSMENT 4.



ANSWERS TO LEARNING ACTIVITIES 12.3.1.to.12.3.4

Learning activity 12.3.1.1

1. a and d

2. (a) Domain: 2, 3, 0, -3
Range: -1, 4, 2, -2(b) Domain: -4, 0, 3, 1, -3
Range: 0, -2, 5, -3

3. (a) $f(2) = 3(2) - 1 = 6 - 1 = 5$
 (b) $f(0) = 3(0) - 1 = 0 - 1 = -1$
 (c) $f(x+2) = 3(x+2) - 1 = 3x + 6 - 1 = 3x + 5$

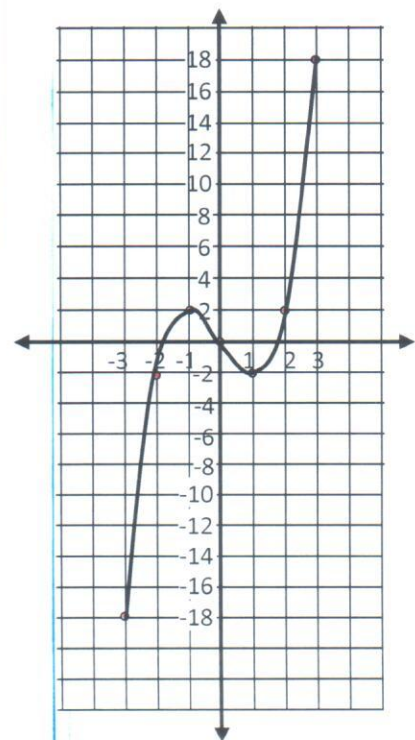
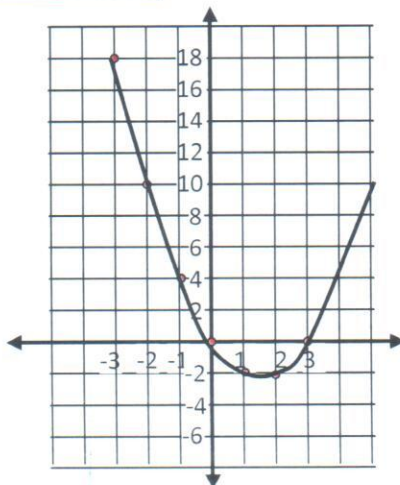
$$4. \frac{f(5) - f(2)}{6} = \frac{[2(5)^2 - 3] - [2(2)^2 - 3]}{6} = \frac{47 - 5}{6} = \frac{42}{6} = 7$$

5. (a)

x	$y = x^2 - 3x$
3	0
2	-2
1	-1
0	0
-1	4
-2	10
-3	18

(b)

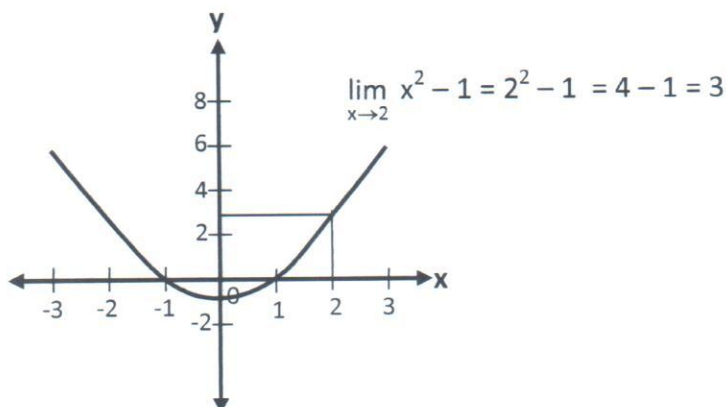
x	$y = x^3 - 3x$
1	-2
2	2
3	18
0	0
-1	2
-2	-2
-3	-18





Learning activity 12.3.1.2

1.



2. (a) $\lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x - 1} = \frac{1^2 - 2(1) + 1}{1 - 1} = \frac{1 - 2 + 1}{0} = \frac{0}{0} = 0$

(b) $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 - 2x - 15} = \frac{x + 3}{(x + 3)(x - 5)} = \frac{1}{x - 5} = \frac{1}{-3 - 5} = -\frac{1}{8}$

(c) $\lim_{x \rightarrow -3} \frac{x^2 - 4}{x^2 - 8x + 12} = \frac{(x + 2)(x - 2)}{(x + 2)(x - 6)} = \frac{x - 2}{x - 6} = \frac{-3 - 2}{-3 - 6} = \frac{-5}{-9} = \frac{5}{9}$

(d) $\lim_{t \rightarrow -2} x^2 + 2x + 4 = -2^2 + 2(-2) + 4 = 4 - 4 + 4 = 4$

(e) $\lim_{t \rightarrow 8} P = 12t^2 - t^4 + 5$
 $= 12(8)^2 - 8^4 + 5$
 $= 12(64) - 4096 + 5$
 $= 768 - 4091$
 $= -3323$

Learning activity 12.3.1.3

- | | |
|--------|--------------------|
| 1. 5 | 5. $-\frac{1}{2}$ |
| 2. 4 | 6. 4 |
| 3. -4 | 7. $3a^2 + 4a - 5$ |
| 4. 289 | 8. 5 |

Learning activity 12.3.1.4

- | | | | |
|-------|-------------------|--------------------------|------------------|
| 1. 14 | 2. $\frac{16}{7}$ | 3. $\frac{1}{2\sqrt{2}}$ | 4. $\frac{1}{2}$ |
|-------|-------------------|--------------------------|------------------|

**Learning activity 12.3.1.5**

1. Solution:

- (i) To find
- $\lim_{x \rightarrow 0^-} f(x)$
- , we consider values of
- f
- when
- $x < 0$
- .

$$\text{Thus } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} f(-1) = -1.$$

- (ii) To find
- $\lim_{x \rightarrow 0^+} f(x)$
- , we consider values of
- f
- when
- $x > 0$
- .

$$\text{Thus } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1) = 1.$$

- (ii) To find
- $\lim_{x \rightarrow 0} f(x)$
- , since
- $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$
- ,

$$\text{Then } \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

2. Solution:

- (i) To find
- $\lim_{x \rightarrow 3^-} f(x)$
- , we consider values of
- f
- when
- $x < 3$
- , i.e.

$$f(x) = x - 1.$$

$$\text{Thus, } \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x - 1) = \lim_{x \rightarrow 3^-} x - \lim_{x \rightarrow 3^-} 1 = 3 - 1 = 2$$

- (ii) To find
- $\lim_{x \rightarrow 3^+} f(x)$
- , we consider values of
- f
- when
- $x > 3$
- , i.e.

$$f(x) = x^2 - 7.$$

$$\text{Thus, } \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x^2 - 7) = 3^2 - 7 = 9 - 2 = 2$$

- (iii) Since
- $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = 2$
- , then
- $\lim_{x \rightarrow 3} f(x) = 2$

Learning activity 12.3.1.6

1. $\lim_{x \rightarrow 0^+} \frac{3}{x^3} = +\infty$
2. $\lim_{x \rightarrow 1^+} \frac{4(x-1)}{x^2-1} = +\infty$
3. $\lim_{x \rightarrow 3^-} \frac{x-3}{x^2-6x+9} = -\infty$
4. $\lim_{x \rightarrow 0^+} \frac{x^2+3x-3}{x} = +\infty$
5. $\lim_{x \rightarrow 3^-} \frac{x^2-9}{x^3-27} = \frac{2}{9}$

**Learning activity 12.3.1.7**

1. $\lim_{x \rightarrow +\infty} \frac{1}{x^8} = 0$
2. $\lim_{x \rightarrow +\infty} \frac{12}{x^{15}} = 0$
3. $\lim_{x \rightarrow +\infty} \frac{x^2 + 1}{1 - x^2} = -1$
4. $\lim_{x \rightarrow -\infty} \frac{3x^3 + 2x + 1}{x^3 + 3x + 2} = 3$
5. $\lim_{x \rightarrow +\infty} 3 + \frac{1}{x} + \frac{2}{x^2} = 3$

Learning activity 12.3.1.8

1. Function f is defined at $x = 1$.
 - i). When $x = 1$, we use $f(x) = 2$, thus $f(1) = 2$
 - ii.) $\lim_{x \rightarrow 1} (3x - 5) = 3(1) - 5 = -2$,
 - But iii.) $\lim_{x \rightarrow 1} f(x) \neq f(1)$
 so condition iii.) is not satisfied and function f is NOT continuous at $x = 1$.
2. Function f is defined at $x = -1$ since it leads to division of zero. Thus $f(-1)$ is indeterminate or does not exist.
Hence, the function $f(x)$ is NOT continuous at $x = -1$.

Summative Task 1**A. Multiple Choice**

- | | |
|------|-------|
| 1. D | 6. B |
| 2. B | 7. D |
| 3. D | 8. C |
| 4. A | 9. D |
| 5. D | 10. D |

- B. 1. -1 2. 2 3. 3

Learning activity 12.3.2.1

- | | |
|--|--|
| 1. $6x + 2$ | 5. Tangent line : $y - 9 = 6(x - 3)$ |
| 2. $y + 4x + 8 = 0$ | Normal line: $y - 9 = -\frac{1}{6}(x - 3)$ |
| 3. $m_{\text{tan}} = -1$ | |
| 4. $m_{\text{tan}} = 14$ | |
| $y - 17 = 14(x - 2)$ or $y = 14x - 11$ | |

**Learning activity 12.3.2.2**

1. $\frac{dy}{dx} = 5x^4$

2. $\frac{dy}{dx} = 3x^2 - 4x + 4$

3. $\frac{dy}{dx} = 4x^3 - 6x^2 + 3$

4. $\frac{dy}{dw} = \frac{6}{(w-4)^2}$

5. $\frac{dy}{dx} = 2x$

6. $b = -3; c = 2, d = 1$

Learning activity 12.3.2.3

1. $\frac{dy}{dx} = 4(3x^2 - 4x + 5)^3(6x - 4)$

2. $\frac{dy}{dx} = \frac{2}{3}(2x^3 - 4x + 1)^{\frac{1}{3}}(6x - 4)$

3. $\frac{dy}{dx} = \frac{(x+1)^2 - 2(x^2 - 1)}{(x+1)^4}$

4. $\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}} - \frac{1}{2}x(x+1)^{-\frac{1}{2}}}{(x+1)}$

Learning activity 12.3.2.4

1. $\frac{d^2y}{dx^2} = 12$

2. $\frac{d^2y}{dx^2} = 3x^2 - 2x + 1$

3. $\frac{d^2y}{dx^2} = 6x$

4. $\frac{d^2y}{dx^2} = \frac{13}{3x^3}$

Learning activity 12.3.2.5

1. a. 9 m/sec
b. at $t = 1$ sec or $t = 3$ sec
c. before 1 sec or after 3 sec

2. a. 61.22 sec b. 299.76 m/sec c. 30.6 sec d. -9.8 m/sec

3. a. $s = 490$ m b. 980 m/sec

Learning activity 12.3.2.6

1. 12 ft/sec
2. 128π m²/sec
3. 0.162 m/sec
4. $\frac{3}{\pi}$ ft/min

**Learning activity 12.3.2.7**

1. $\frac{dy}{dx} = 10 \cos x$
2. $\frac{dy}{dx} = 8 \cos(2x)$
3. $\frac{dy}{dx} = 14 \sec^2 x \cos x - 14 \tan x \sin x - 1 \csc x \cot x$
4. $\frac{dy}{dx} = \frac{x^4(5 \sin x - x \cos x)}{4 \sin^2 x}$
5. $\frac{dy}{dx} = 200 \sec^2(20x)$

Summative Task 2

- A. 1. A 2. D 3. B 4. D 5. A
6. A 7. B 8. A 9. A 10. C
11. $6x^2 \cos 2x^3$
12. $15x^2(\sec^2 5x^3)$
13. $20x^4 \sec 4x^5(\tan 4x^5)$
14. $15x^4(\sin^2 x^5 \cos x^5)$
15. $-25x^4 \csc 5x^5(\cot 5x^5)$

Learning activity 12.3.3.1

- A. 1. (3, 0)
2. (1, 2)(-1, 6)
3. (-2, 0)
- B. 1. Absolute minimum: (-3, 3)
Absolute maximum: (-1, 7)
2. Absolute minimum: (2, -7)
No absolute maxima.

Learning activity 12.3.3.2

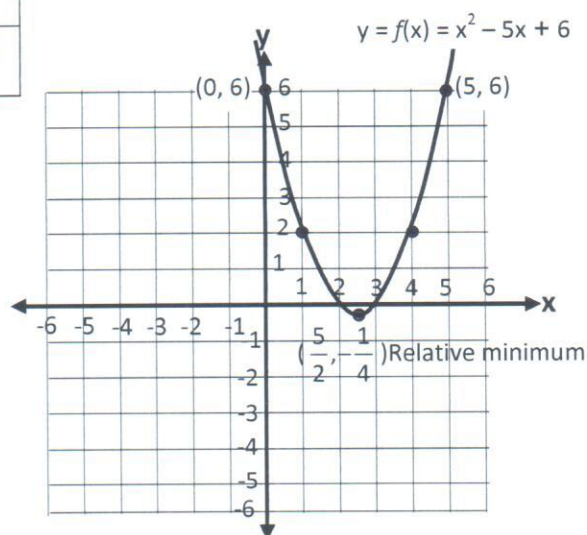
1. 50, 50
2. The largest area is 46 656 sq. units
3. The maximum area is 80 000 m²
4. 125 m by 250 m



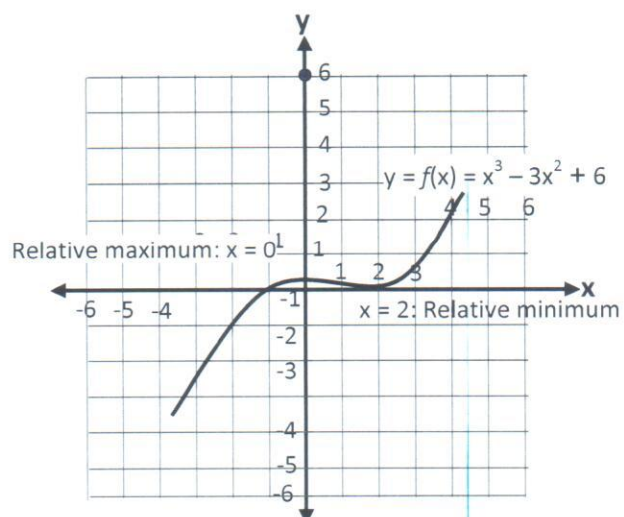
Learning activity 12.3.3.3

1. $y = f(x) = x^2 - 5x + 6$

Interval	x	f(x)	f'(x)	Conclusion
$x < \frac{5}{2}$	2	0	+	f is increasing
$x = \frac{5}{2}$	$\frac{5}{2}$	$-\frac{1}{4}$	0	Rel. minimum
$x > \frac{5}{2}$	3	0	+	f is increasing



2. $y = f(x) = x^3 - 3x^2 + 6$.
- Increasing on $(-\infty, 0)$ and $(2, \infty)$
 - Decreasing on $(0, 2)$
 - relative maximum at $x = 0$
 - relative minimum at $x = 2$
 - Sketch the graph.

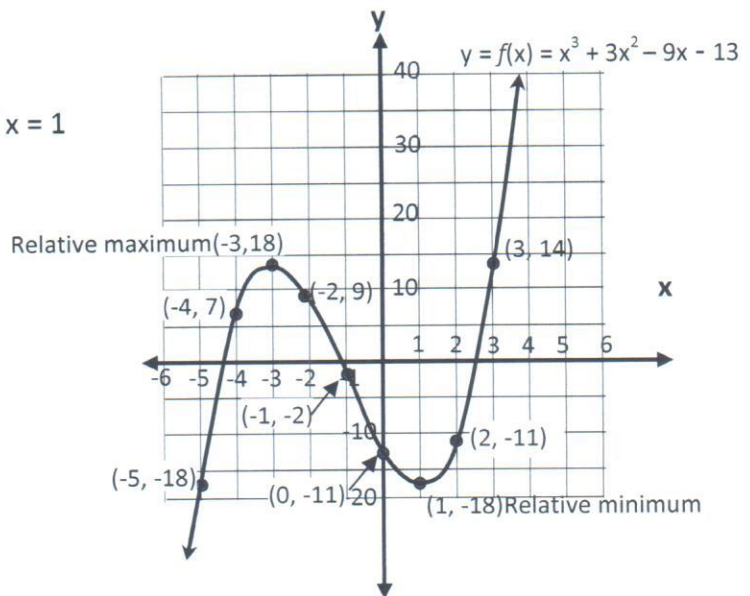




Learning activity 12.3.3.4

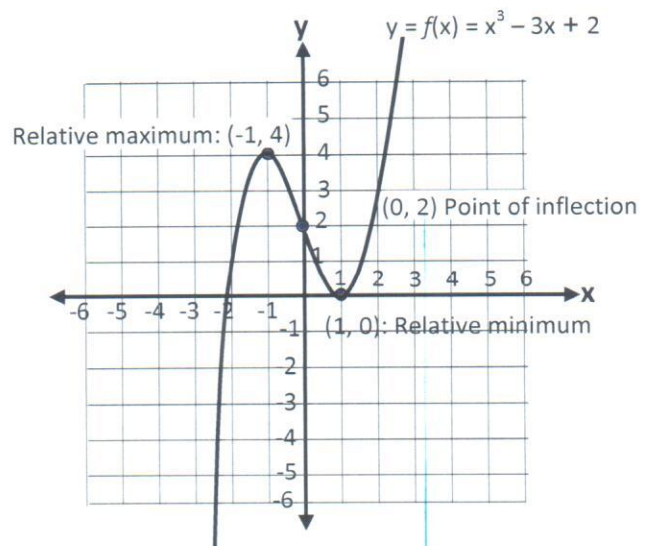
1. $f(x) = x^3 + 3x^2 - 9x - 13$

- Critical numbers: $x = -3$ and $x = 1$
- Relative maximum: $(-3, 14)$
- Relative minimum: $(1, -18)$
- Sketch the graph



2. $f(x) = x^3 - 3x + 2$

- Critical numbers: $x = 1$ and $x = -1$
- Relative maximum: $(-1, 4)$
- Relative minimum: $(1, 0)$
- Points of inflection: $(0, 2)$
- Concavity: f is concave down on the interval $(-\infty, 0)$ and concave up on $(0, \infty)$.
- Sketch the graph of f .



Learning activity 12.3.3.5

- $x = -9$, $y = 9$; product is -81 .
- $x = 8$, $y = 24$; the maximum product is 192
- The distance is minimum if the vertex of the right triangle is at the middle of the wire.

**Learning activity 12.3.3.6**

1. $dy = (2x + 7)dx$
2. $dy = (8x^3 - 6x^2)dx$
3. $dy = 600, \Delta y = 603$
4. 4.00125
5. 20π cu.cm

Summative Task 3

- A.
- | | | | |
|----|---|-----|---|
| 1. | B | 6. | B |
| 2. | A | 7. | D |
| 3. | B | 8. | A |
| 4. | C | 9. | C |
| 5. | C | 10. | D |
- B.
1. The length of the side of the square to be cut is 2 cm so that the box will have the largest possible volume.

2.

Interval	$x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$x > 3$
x	0	1	2	3	4
Value of $f(x)$	2	0	1	2	0
Value of $f'(x)$	-	0	+	0	-
Conclusion	f is decreasing	Relative minimum	f is increasing	Relative maximum	f is decreasing

Learning activity 12.3.4.1

1. $x^2 + 3x + C$
2. $\frac{x^3}{3} + \frac{3x^2}{2} + 2x + Cx^2$
3. $\frac{x^4}{4} - \frac{x^2}{2} + C$
4. $16x^4 - 48x^3 + 54x^2 - 27x + C$

Learning activity 12.3.4.2

- | | | | | |
|--------------------|--------|-------|-------|------|
| 1. $\frac{140}{3}$ | 2. 104 | 3. 44 | 4. 16 | 5. 0 |
|--------------------|--------|-------|-------|------|



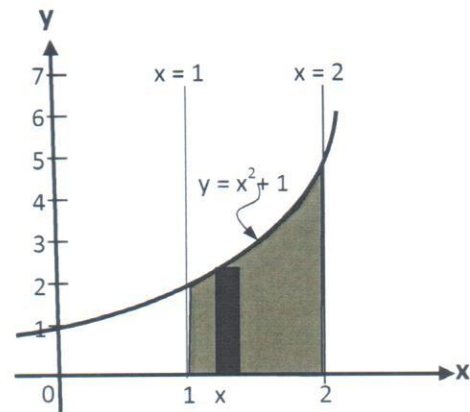
Learning activity 12.3.4.3

1.

$$A = \int_1^2 (x^2 + 1) dx$$

$$A = \left[\frac{x^3}{3} + x \right]_1^2 = \left[\frac{8}{3} + 2 \right] - \left[\frac{1}{3} + 1 \right]$$

$$A = \frac{14}{3} - \frac{4}{3} = \frac{10}{3} \text{ sq.unit}$$



2.

$$A = \int_1^2 [(x^2 + 1) - x] dx = \int_1^2 [(x^2 + 1) - x] dx$$

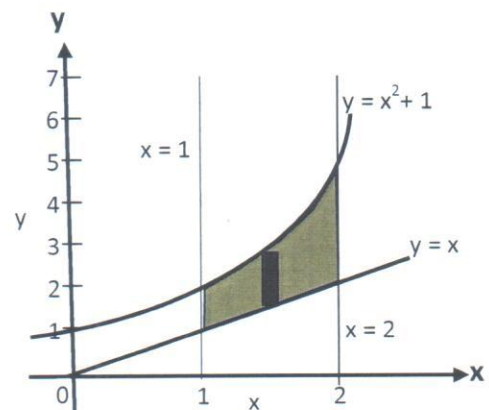
$$A = \left[\frac{x^3}{3} + x \right]_1^2 - \left[\frac{x^2}{2} \right]_1^2$$

$$A = - \left[\left(\frac{2^3}{3} + 2 \right) - \left(\frac{1^3}{3} + 1 \right) \right] - \left[\frac{2^2}{2} - \frac{1^2}{2} \right]$$

$$A = \left[\left(\frac{8}{3} + 2 \right) - \frac{4}{3} \right] - \left[\frac{4}{2} - \frac{1}{2} \right]$$

$$A = \left[\frac{14}{3} - \frac{4}{3} \right] - \left[\frac{3}{2} \right]$$

$$A = \frac{10}{3} - \frac{3}{2} = \frac{20}{6} - \frac{9}{6} = \frac{11}{6} \text{ sq.units}$$



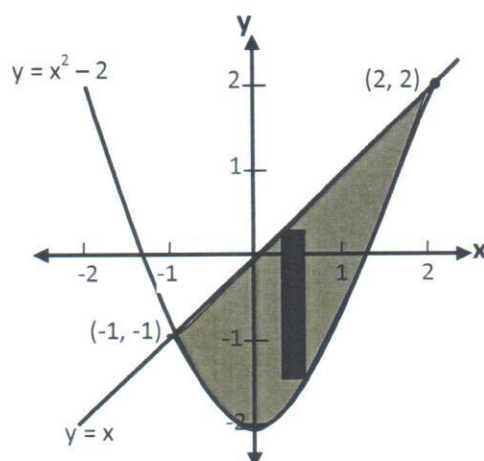
Summative Task 4

A. 1. C 2. C 3. C 4. A 5. D

- B. 1. $x^2 + 3x + C$
2. $\frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$
3. $\frac{(x^2 + 3x + 2)^4}{4} + C$
4. 16



- C. Area of the region = $\frac{9}{2}$ sq. units



END OF UNIT MODULE 12.3



REFERENCES

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SUBJECT AND GRADE TO STUDY

Grade Levels	Subjects
Grades 7 and 8	1. English
	2. Mathematics
	3. Science
	4. Social Science
	5. Making a Living
	6. Personal Development
	7. English
Grades 9 and 10	1. English
	2. Formal Mathematics
	3. Practical Mathematics
	4. Science
	5. Social Science
	6. Commerce
	7. Design and Technology- Computing
	8. Personal Development
Grades 11 and 12	1. English
	• Applied English
	• Language and Literature
	2. Mathematics
	• Mathematics A
	• Mathematics B
	3. Science
	4. Social Science
	• History
	• Geography
	• Economics
	5. Business Studies
	6. Personal Development
	7. ICT

REMEMBER:

- For Grades 7 and 8, you are required to do all six (6) courses.
- For Grades 9 and 10, you must study English, Mathematics, Science, Personal Development, Social Science and Commerce. Design and Technology-Computing is optional.
- For Grades 11 and 12, you are required to complete seven (7) out of thirteen (13) courses to be certified. Your Provincial Coordinator or Supervisor will give you more information regarding each subject.

Certificate in Matriculation

CORE COURSES

Basic English
English 1
English 2
Basic Maths
Maths 1

Maths 2

History of Science & Technology

OPTIONAL COURSES

Science Streams: Biology

Chemistry, Physics and Social Science Streams:

Geography, Introduction to Economics and Asia and the Modern World

REMEMBER:

You must successfully complete 8 courses; 5 compulsory and 3 optional

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